THE 3-RAINBOW INDEX OF A GRAPH

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Abstract

Let $G$ be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A tree $T$ in $G$ is a rainbow tree if no two edges of $T$ receive the same color. For a vertex subset $S \subseteq V(G)$, a tree that connects $S$ in $G$ is called an $S$-tree.

The minimum number of colors that are needed in an edge-coloring of $G$ such that there is a rainbow $S$-tree for each $k$-subset $S$ of $V(G)$ is called the $k$-rainbow index of $G$, denoted by $rx_k(G)$. In this paper, we first determine the graphs of size $m$ whose 3-rainbow index equals $m$, $m - 1$, $m - 2$ or 2. We also obtain the exact values of $rx_3(G)$ when $G$ is a regular multipartite complete graph or a wheel. Finally, we give a sharp upper bound for $rx_3(G)$ when $G$ is 2-connected and 2-edge connected. Graphs $G$ for which $rx_3(G)$ attains this upper bound are determined.

Keywords: rainbow tree, $S$-tree, $k$-rainbow index.

2010 Mathematics Subject Classification: 05C05, 05C15, 05C75.

1. Introduction

We follow the terminology and notation of Bondy and Murty [1]. Let $G$ be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A path of $G$ is a rainbow path if

¹Supported by NSFC No.11371205 and 11071130.
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no two edges of the path are colored the same. The graph \( G \) is *rainbow connected* if for every two vertices \( u \) and \( v \) of \( G \), there is a rainbow path connecting \( u \) and \( v \). The minimum number of colors for which there is an edge-coloring of \( G \) such that \( G \) is rainbow connected is called the *rainbow connection number* of \( G \), denoted by \( rc(G) \). Results on the rainbow connections can be found in \([2, 3, 5, 6, 7]\).

These concepts were introduced by Chartrand et al. in \([3]\). In \([4]\), they generalized the concept of rainbow path to rainbow tree. A tree \( T \) in \( G \) is a *rainbow tree* if no two edges of \( T \) receive the same color. For \( S \subseteq V(G) \), a *rainbow \( S \)-tree* is a rainbow tree that connects the vertices of \( S \). Given a fixed integer \( k \) with \( 2 \leq k \leq n \), the edge-coloring \( c \) of \( G \) is called a *\( k \)-rainbow coloring* if for every \( k \)-subset \( S \) of \( V(G) \), there exists a rainbow \( S \)-tree. In this case, \( G \) is called *\( k \)-rainbow connected*. The minimum number of colors that are needed in a \( k \)-rainbow coloring of \( G \) is called the *\( k \)-rainbow index* of \( G \), denoted by \( rx_k(G) \).

Clearly, when \( k = 2 \), \( rx_2(G) \) is the rainbow connection number \( rc(G) \) of \( G \). For every connected graph \( G \) of order \( n \), it is easy to see that \( rx_2(G) \) is less than or equal to \( rx_3(G) \), which is less than or equal to \( \cdots \leq rx_n(G) \).

The *Steiner distance* \( d(S) \) of a subset \( S \) of vertices in \( G \) is the minimum size of a tree in \( G \) that connects \( S \). Such a tree is called a *Steiner \( S \)-tree* or simply a *Steiner tree*. The *\( k \)-Steiner diameter*, \( sdiam_k(G) \), of \( G \) is the maximum Steiner distance of \( S \) among all \( k \)-subsets \( S \) of \( G \). Then there is a simple upper bound and a lower bound for \( rx_k(G) \).

**Observation 1.1** \([4]\). For every connected graph \( G \) of order \( n \geq 3 \) and each integer \( k \), with \( 3 \leq k \leq n \), \( k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1 \).

It was shown in \([4]\) that trees are contained in a class of graphs whose \( k \)-rainbow index attains the upper bound.

**Proposition 1.2** \([4]\). Let \( T \) be a tree of order \( n \geq 3 \). For each integer \( k \), with \( 3 \leq k \leq n \), \( rx_k(T) = n - 1 \).

The authors of \([4]\) also gave the following observation.

**Observation 1.3** \([4]\). Let \( G \) be a connected graph of order \( n \) containing two bridges \( e \) and \( f \). For each integer \( k \) with \( 2 \leq k \leq n \), every \( k \)-rainbow coloring of \( G \) must assign distinct colors to \( e \) and \( f \).

For \( k = 2 \), \( rx_2(G) = rc(G) \), this case has been studied extensively, see \([6, 7]\). But for \( k \geq 3 \), very few results has been obtained. In this paper, we focus on \( k = 3 \). By Observation 1.1, we have \( rx_3(G) \geq 2 \). On the other hand, if \( G \) is a nontrivial connected graph of size \( m \), then the coloring that assigns distinct colors to the edges of \( G \) is a 3-rainbow coloring, hence \( rx_3(G) \leq m \). So we want to determine the graphs whose 3-rainbow index equals the values \( m \), \( m - 1 \), \( m - 2 \) and 2, respectively. The following results are needed.
Lemma 1.4 [4]. For $3 \leq n \leq 5$, $rx_3(K_n) = 2$.

Lemma 1.5 [4]. Let $G$ be a connected graph of order $n \geq 6$. For each integer $k$ with $3 \leq k \leq n$, $rx_k(G) \geq 3$.

Theorem 1.6 [4]. For each integer $k$ and $n$ with $3 \leq k \leq n$,

$$rx_k(C_n) = \begin{cases} n - 2, & \text{if } k = 3 \text{ and } n \geq 4, \\ n - 1, & \text{if } k = n = 3 \text{ or } 4 \leq k \leq n. \end{cases}$$

Theorem 1.7 [4]. If $G$ is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$, then

$$rx_k(G) = \begin{cases} n - 2, & \text{if } k = 3 \text{ and } g \geq 4, \\ n - 1, & \text{if } g = 3 \text{ or } 4 \leq k \leq n. \end{cases}$$

The following observation is easy to verify.

Observation 1.8. Let $G$ be a connected graph and $H$ be a connected spanning subgraph of $G$. Then $rx_3(G) \leq rx_3(H)$.

In Section 2, we determine the graphs whose 3-rainbow index equals the values $m, m - 1, m - 2$ or 2. In Section 3, we determine the 3-rainbow index for the complete bipartite graphs $K_{r,r}$ and complete $t$-partite graphs $K_t \times r$ as well as the wheel $W_n$. Finally, we give a sharp upper bound of $rx_3(G)$ for 2-connected graphs and 2-edge connected graphs. Moreover, graphs whose 3-rainbow index attains the upper bound are characterized.

2. Graphs with $rx_3(G) = m, m - 1, m - 2$ or 2

At first, we consider the graphs with $rx_3(G) = 2$. From Lemma 1.5, if $rx_3(G) = 2$, then the order $n$ of $G$ satisfies $3 \leq n \leq 5$.

Theorem 2.1. Let $G$ be a connected graph of order $n$. Then $rx_3(G) = 2$ if and only if $G = K_5$ or $G$ is a 2-connected graph of order 4 or $G$ is of order 3.

Proof. If $n = 3$, then it is easy to see that $rx_3(G) = 2$.

If $n = 4$, assume that $G$ is not 2-connected, then there is a cut vertex $v$. It is easy to see that a tree connecting the vertices of $G - v$ has size 3, thus $rx_3(G) \geq 3$, a contradiction.

If $n = 5$, then let $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$. Assume that $rx_3(G) = 2$ but $G$ is not $K_5$. Let $c : E(G) \to \{1, 2\}$ be a rainbow coloring of $G$. Since every three vertices belong to a rainbow path of length 2, there is no monochromatic triangle. Now we show that the maximum degree $\Delta(G)$ is 4. If $\Delta(G)$ is 2, then $G$ is a cycle or a path, and it is easy to check that $rx_3(G)$ is 3 or 4, a contradiction.

If $\Delta(G) = 4$, then there are two vertices $x, y$ with degree 2 that have the same neighbors $u, v, w$. Let $c(x) = c(u) = 1$, $c(x) = 2$, and $c(y) = c(v) = c(w) = 1$. Then $c(x)$ has two neighbors with the same color, a contradiction.
Assume that $\Delta(G)$ is 3. Let $\deg(v_1) = 3$ and $N(v_1) = \{v_2, v_3, v_4\}$. Then at least two edges incident to $v_1$ have the same color, say $c(v_1v_2) = c(v_1v_3) = 1$. Consider $\{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}$, this forces $c(v_2v_3) = c(v_3v_4) = 2$. Consider $\{v_1, v_2, v_3\}$, it implies that $c(v_2v_3) = 2$, but now $\{v_2, v_3, v_4\}$ forms a monochromatic triangle, a contradiction. Thus $\Delta(G) = 4$. Suppose $\deg(v_1) = 4$. If there are three edges incident to $v_1$ colored the same, say $c(v_1v_2) = c(v_1v_3) = c(v_1v_4) = 1$, then consider the three vertices $v_2, v_3$ and $v_4$. Since these three vertices must belong to a rainbow path of length 2, without loss of generality, assume that $c(v_2v_3) = 1$ and $c(v_3v_4) = 2$. However then $\{v_1, v_2, v_3\}$ is a monochromatic triangle, which is impossible. Therefore only two edges incident to $v_1$ are assigned the same color. Since $G$ is not $K_5$, $G$ is a spanning subgraph of $K_5 - e$. Since $\deg(v_1) = 4$, we may assume that $G$ is a spanning subgraph of $K_5 - v_1v_4$. Let $G' = K_5 - v_1v_4$. Consider $\{v_1, v_3, v_4\}$, it implies $v_1v_3$ and $v_1v_4$ must have different colors, without loss of generality, assume that $c(v_1v_3) = 1$ and $c(v_1v_4) = 2$. By symmetry, suppose $c(v_1v_2) = 1$ and $c(v_1v_3) = 2$. Then $c(v_2v_3) = 2$, $c(v_4v_5) = 1$. Consider $\{v_2, v_3, v_4\}, \{v_3, v_4, v_5\}, \{v_2, v_3, v_5\}$; then $c(v_2v_4) = 1, c(v_3v_5) = 2, c(v_2v_5) = 1$, but now $\{v_2, v_4, v_5\}$ forms a monochromatic triangle, which is impossible. Hence, $rx_3(G) \geq rx_3(G') \geq 3$, contradicting the assumption.

**Theorem 2.2.** Let $G$ be a connected graph of size $m \geq 3$. Then

(1) $rx_3(G) = m$ if and only if $G$ is a tree.

(2) $rx_3(G) = m - 1$ if and only if $G$ is a unicyclic graph with girth 3.

(3) $rx_3(G) = m - 2$ if and only if $G$ is a unicyclic graph with girth at least 4.

**Proof.** (1) By Proposition 1.2, if $G$ is a tree, then $rx_3(G) = n - 1 = m$. Conversely, if $rx_3(G) = m$ but $G$ is not a tree, then $m \geq n$. By Observation 1.1, $rx_3(G) \leq n - 1 \leq m - 1$, a contradiction.

(2) If $G$ is a unicyclic graph with girth 3, then by Theorem 1.7, $rx_3(G) = n - 1 = m - 1$. Conversely, if $rx_3(G) = m - 1$, then by (1), $G$ must contain cycles. If $G$ contains at least two cycles, then $m \geq n + 1$. By Observation 1.1, $rx_3(G) \leq n - 1 \leq m - 2$, a contradiction. Thus, $G$ contains exactly one cycle. If the cycle of $G$ is of length at least 4, then by Theorem 1.7, $rx_3(G) = n - 2 = m - 2$, a contradiction. Thus, the cycle of $G$ is of length 3, the result holds.

(3) If $G$ is a unicyclic graph with girth at least 4, then by Theorem 1.7, $rx_3(G) = n - 2 = m - 2$. Conversely, if $rx_3(G) = m - 2$ and $m \geq n + 2$, then by Observation 1.1, $rx_3(G) \leq n - 1 \leq m - 3$, a contradiction. Thus, $m \leq n + 1$. If $m = n$, then $G$ is a unicyclic graph. By Theorem 1.7, the girth of $G$ is at least 4. If $m = n + 1$, and there are two edge-disjoint cycles $C_1$ and $C_2$ of lengths, respectively $g_1$ and $g_2$ such that $g_1 \geq g_2$, then if $g_1 \geq 4$, we assign $g_1 - 2$ colors to $C_1$, $g_2 - 1$ new colors to $C_2$ and assign new distinct colors to all the remaining edges, which make $G$ 3-rainbow connected, hence $rx_3(G) \leq m - 3$, a
contradiction. Therefore \( g_1 = g_2 = 3 \). In this case, we assign to each cycle three colors 1, 2, 3, and assign new colors to all the remaining edges. It follows that, then \( G \) is 3-rainbow connected, thus \( rx_3(G) \leq m - 3 \). If these two cycles are not edge-disjoint, we can also use \( m - 3 \) colors to make \( G \) 3-rainbow connected, a contradiction.

3. The 3-rainbow Index of Some Special Graphs

In this section, we determine the 3-rainbow index of some special graphs. First, we consider the regular complete bipartite graphs \( K_{r,r} \). It is easy to see that when \( r = 2 \), \( rx_3(K_{2,2}) = 2 \) and, logically, we can define \( rx_3(K_{1,1}) = 0 \).

**Theorem 3.1.** For each integer \( r \) with \( r \geq 3 \), \( rx_3(K_{r,r}) = 3 \).

**Proof.** Let \( U \) and \( W \) be the partite sets of \( K_{r,r} \), where \( |U| = |W| = r \). Suppose that \( U = \{u_1, \ldots, u_r\} \) and \( W = \{w_1, \ldots, w_r\} \). If \( S \subseteq U \) and \( |S| = 3 \), then every \( S \)-tree has size at least 3; hence \( rx_3(K_{r,r}) \geq 3 \).

Next we show that \( rx_3(K_{r,r}) \leq 3 \). We define a coloring \( c : E(K_{r,r}) \to \{1, 2, 3\} \) as follows.

\[
c(u_iw_j) = \begin{cases} 
1, & \text{if } 1 \leq i = j \leq r, \\
2, & \text{if } 1 \leq i < j \leq r, \\
3, & \text{if } 1 \leq j < i \leq r.
\end{cases}
\]

Now we show that \( c \) is a 3-rainbow coloring of \( K_{r,r} \). Let \( S \) be a set of three vertices of \( K_{r,r} \). We consider two cases.

**Case 1.** The vertices of \( S \) belong to the same partite set of \( K_{r,r} \). Without loss of generality, let \( S = \{u_i, u_j, u_k\} \), where \( i < j < k \). Then \( T = \{u_iw_j, u_jw_j, u_kw_j\} \) is a rainbow \( S \)-tree.

**Case 2.** The vertices of \( S \) belong to different partite sets of \( K_{r,r} \). Without loss of generality, let \( S = \{u_i, u_j, w_k\} \), where \( i < j \).

**Subcase 2.1.** Let \( k < i < j \). Then \( T = \{u_iw_k, u_iw_j, u_jw_j\} \) is a rainbow \( S \)-tree.

**Subcase 2.2.** Let \( i \leq k \leq j \). Then \( T = \{u_iw_k, u_jw_k\} \) is a rainbow \( S \)-tree.

**Subcase 2.3.** Let \( i < j < k \). Then \( T = \{u_iw_i, u_jw_i, u_jw_k\} \) is a rainbow \( S \)-tree.

With the aid of Theorem 3.1, we are now able to determine the 3-rainbow index of complete \( t \)-partite graph \( K_{t \times r} \). Note that we always have \( t \geq 3 \). When \( r = 1 \), \( rx_3(K_{t \times 1}) = rx_3(K_t) \), which was given in [4].

**Theorem 3.2.** Let \( K_{t \times r} \) be a complete \( t \)-partite graph, where \( r \geq 2 \) and \( t \geq 3 \). Then \( rx_3(K_{t \times r}) = 3 \).
Let $U_1, U_2, \ldots, U_t$ be the $t$ partite sets of $K_{t \times r}$, where $|U_i| = r$. Suppose that $U_i = \{u_{i1}, \ldots, u_{ir}\}$. If $S \subseteq U_i$ and $|S| = 3$, then every $S$-tree has size at least 3, hence $r x_3(K_{r, r}) \geq 3$.

Next we show that $r x_3(K_{t \times r}) \leq 3$. We define a coloring $c : E(K_{t \times r}) \rightarrow \{1, 2, 3\}$ as follows.

$$c(u_{ai}u_{bj}) = \begin{cases} 1, & \text{if } 1 \leq i = j \leq r, \\ 2, & \text{if } 1 \leq i < j \leq r, \\ 3, & \text{if } 1 \leq j < i \leq r, \end{cases}$$

where $1 \leq a < b \leq t$.

We now show that $c$ is a 3-rainbow coloring of $K_{t \times r}$. Let $S$ be a set of three vertices of $K_{t \times r}$.

**Case 1.** The vertices of $S$ belong to the same partite set. Without loss of generality, let $S = \{u_{a1}, u_{a2}, u_{a3}\}$. Then $T = \{u_{a1}u_{b2}, u_{a2}u_{b2}, u_{a3}u_{b2}\}$ is a rainbow $S$-tree.

**Case 2.** Two vertices of $S$ belong to the same partite set. Without loss of generality, let $S = \{u_{ai}, u_{aj}, u_{bk}\}$. If $k < i < j$, then $T = \{u_{ai}u_{bk}, u_{ai}u_{bj}, u_{aj}u_{bj}\}$ is a rainbow $S$-tree. If $i < j < k$, then $T = \{u_{ai}u_{bi}, u_{aj}u_{bi}, u_{aj}u_{bk}\}$ is a rainbow $S$-tree.

**Case 3.** Each vertex of $S$ belongs to a different partite set. Let $S = \{u_{ai}, u_{bj}, u_{ck}\}$, $a < b < c$.

**Subcase 3.1.** Assume that $i = j = k$. Without loss of generality, let $S = \{u_{ai}, u_{b1}, u_{c1}\}$. Then $T = \{u_{ai}u_{b1}, u_{ai}u_{b2}, u_{b2}u_{c1}\}$ is a rainbow $S$-tree.

**Subcase 3.2.** Suppose that $i = j \neq k$. Without loss of generality, let $S = \{u_{ai}, u_{b1}, u_{c2}\}$. Clearly, $T = \{u_{ai}u_{b1}, u_{b1}u_{c2}\}$ is a rainbow $S$-tree.

**Subcase 3.3.** Let $i \neq j \neq k$. Without loss of generality, let $S = \{u_{a1}, u_{b2}, u_{c3}\}$. Then $T = \{u_{a1}u_{c1}, u_{c1}u_{b2}, u_{b2}u_{c3}\}$ is a rainbow $S$-tree.

Another well-known class of graphs are wheels. For $n \geq 3$, the wheel $W_n$ is a graph constructed by joining a vertex $v$ to every vertex of a cycle $C_n : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$. Given an edge-coloring $c$ of $W_n$, for two adjacent vertices $v_i$ and $v_{i+1}$, we define an edge-coloring of the graph by identifying $v_i$ and $v_{i+1}$ to a new vertex $v'$ as follows: set $c(vv') = c(vv_{i+1})$, $c(v_{i-1}v') = c(v_{i-1}v_i)$, $c(v'v_{i+2}) = c(v_{i+1}v_{i+2})$, and keep the coloring for the remaining edges. We call this coloring the **identified-coloring** at $v_i$ and $v_{i+1}$. Next we determine the 3-rainbow index of wheels.
Theorem 3.3. For $n \geq 3$, the 3-rainbow index of the wheel $W_n$ is

$$rx_3(W_n) = \begin{cases} 
2, & \text{if } n = 3, \\
3, & \text{if } 4 \leq n \leq 6, \\
4, & \text{if } 7 \leq n \leq 16, \\
5, & \text{if } n \geq 17.
\end{cases}$$

Proof. Suppose that $W_n$ consists of a cycle $C_n : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ and another vertex $v$ joined to every vertex of $C_n$.

Since $W_3 = K_4$, it follows by Lemma 1.4 that $rx_3(W_3) = 2$.

If $n = 6$, then let $S = \{v_1, v_2, v_4\}$. Since every $S$-tree has size at least 3, $rx_3(W_6) \geq 3$. Next we show that $rx_3(W_6) \leq 3$ by providing a 3-rainbow coloring of $W_6$ as follows:

$$c(e) = \begin{cases} 
1, & \text{if } e \in \{uv_1, uv_4, v_2v_3, v_5v_6\}, \\
2, & \text{if } e \in \{vv_2, vv_5, v_3v_4, v_1v_6\}, \\
3, & \text{if } e \in \{v_3, v_6, v_4v_5, v_1v_2\}.
\end{cases}$$

If $n = 5$, then $|V(W_5)| = 6$ and, by Lemma 1.5, $rx_3(W_5) \geq 3$. Then we show that $rx_3(W_5) \leq 3$. We provide a 3-rainbow coloring of $W_5$ obtained from the 3-rainbow coloring of $W_6$ by the identified-coloring at $v_5$ and $v_6$.

If $n = 4$, then by Theorem 2.1, $rx_3(W_4) \geq 3$. Then we show that $rx_3(W_4) \leq 3$. We provide a 3-rainbow coloring of $W_4$ obtained from the 3-rainbow coloring of $W_6$ by the identified-coloring at $v_5$ and $v_6$, $v_4$ and $v_5$, respectively.

Claim 1. If $7 \leq n \leq 16$, then $rx_3(W_n) = 4$.

First we show that $rx_3(W_7) \geq 4$. Assume, to the contrary, that $rx_3(W_7) \leq 3$. Let $c : E(W_7) \to \{1, 2, 3\}$ be a 3-rainbow coloring of $W_7$. Since $deg(v) = 7 > 2 \times 3$, there exists $A \subseteq V(C_n)$ such that $|A| = 3$ and all edges in $\{uv : u \in A\}$ are colored the same. Thus, there must exist at least two vertices $v_i, v_j \in A$ such that $deg_{C_7}(v_i, v_j) \geq 2$ and a vertex $v_k \in C_7$ such that $v_k \notin \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\}$. Let $S = \{v_i, v_j, v_k\}$. Note that the only $S$-tree of size 3 is $T = vv_i \cup vv_j \cup vv_k$, but $c(vv_i) = c(vv_j)$, it follows that there is no rainbow $S$-tree, which is a contradiction.

Similarly, we have $rx_3(W_n) \geq 4$ for all $n \geq 8$.

Second, we show that $rx_3(W_{16}) \leq 4$, which we establish by defining a 4-rainbow coloring of $W_{16}$ as shown in Figure 1. It is easy to check that $c$ is a 4-rainbow coloring of $W_{16}$. Therefore, $rx_3(W_{16}) = 4$.

When $13 \leq n \leq 15$, we obtain a 4-rainbow coloring of $W_{15}$, $W_{14}$, $W_{13}$ from the 4-rainbow coloring $c$ of $W_{16}$ by consecutively using the identified-colorings at $v_1$ and $v_{16}$, $v_{12}$ and $v_{13}$, $v_8$ and $v_9$.

When $n = 12$, we define a 4-rainbow coloring of $W_{12}$ as shown in Figure 1.

When $7 \leq n \leq 11$, we obtain a 4-rainbow coloring of $W_{11}$, $W_{10}$, $W_9$, $W_8$, $W_7$ from the 4-rainbow coloring $c$ of $W_{12}$ by consecutively using the identified-colorings at $v_1$ and $v_2$, $v_4$ and $v_5$, $v_7$ and $v_8$, $v_{10}$ and $v_{11}$, $v_{11}$ and $v_{12}$. 

Let $A$ be a connected and 2-edge-connected graph. We start with some lemmas that will
be useful in our analysis.

**Claim 2.** If $n \geq 17$, then $rx_3(W_n) = 5$.

First we show that $rx_3(W_{17}) \geq 5$. Assume, to the contrary, that $rx_3(W_{17}) \leq 4$.

Let $c : E(W_{17}) \rightarrow \{1, 2, 3, 4\}$ be a 4-rainbow coloring of $W_{17}$. Since $\deg(v) = 17 > 4 \times 4$, there exists $A \subseteq V(C_n)$ such that $|A| = 5$ and all edges in $\{uv : u \in A\}$ are colored the same, say 1. Suppose that $A = \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}\}$, where $i_1 \leq i_2 \leq i_3 \leq i_4 \leq i_5$. There exists $k$ such that $\deg_C(v_{i_k}, v_{i_{k+1}}) \geq 3$, where $1 \leq k \leq 4$.

Let $S = \{v_{i_k}, v_{i_{k+1}}, v_{i_{k+3}}\}$. Since $d_{C_{17}}(v_{i_k}, v_{i_{k+3}}) \geq 2$ and $d_{C_{17}}(v_{i_{k+1}}, v_{i_{k+3}}) \geq 2$, the only possible $S$-tree is the path $P = v_{i_{k+1}}v_{i_{k+2}}v_{i_{k+3}}v_{i_{k+4}}v_{i_{k+5}}$, where addition is performed modulo 5. Thus color 1 must appear in $P$ and every edge of the path must have a distinct color. By symmetry, we consider two cases. First, let $c(v_{i_{k+1}}v_{i_{k+2}}) = 1$. Suppose $c(v_{i_{k+2}}v_{i_{k+3}}) = 2$, $c(v_{i_{k+3}}v_{i_{k+4}}) = 3$. There exists a vertex $v_0$, where $c(vv_0) = 2$ or 3, such that $d(v_0, A) \geq 3$. It is easy to see that there is no rainbow $\{v_0, v_{i_{k+2}}, v_{i_{k+4}}\}$-tree. In the remaining case, if $c(v_{i_{k+2}}v_{i_{k+3}}) = 1$, then we can also find such a vertex $v_0$ such that there exists no $\{v_0, v_{i_{k+2}}, v_{i_{k+3}}\}$-tree, which is a contradiction.

To show that $rx_3(W_n) \leq 5$ for $n \geq 17$, we define a 5-rainbow coloring of $W_n$ as follows:

$$c(e) = \begin{cases} j, & \text{if } e = vv_i \text{ and } i \equiv j \pmod{5}, 1 \leq j \leq 5, \\ i + 3, & \text{if } e = v_iv_{i+1}. \end{cases}$$

It is easy to see that $c$ is a 5-rainbow coloring of $W_n$. Therefore, $rx_3(W_n) = 5$ for $n \geq 17$. $lacksquare$

4. The 3-rainbow Index of 2-connected and 2-edge-connected Graphs

In this section, we give a sharp upper bound of the 3-rainbow index for 2-connected and 2-edge-connected graphs. We start with some lemmas that will
be used in the sequel.

**Lemma 4.1.** Let $G$ be a connected graph and $\{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$. If each $V_i$ induces a connected subgraph $H_i$ of $G$, then $rx_3(G) \leq k - 1 + \sum_{i=1}^{k} rx_3(H_i)$.

**Proof.** Let $G'$ be a graph obtained from $G$ by contracting each $H_i$ to a single vertex. Then $G'$ is a graph of order $k$, so $rx_3(G') \leq k - 1$. Take an edge-coloring of $G'$ with $k - 1$ colors such that $G'$ is 3-rainbow connected. Now go back to $G$, and color each edge connecting vertices in distinct $H_i$ with the color of the corresponding edge in $G'$. For each $i = 1, 2, \ldots, k$, we use $rx_3(H_i)$ new colors to assign the edges of $H_i$ such that $H_i$ is 3-rainbow connected. The resulting edge-coloring makes $G$ 3-rainbow connected. Therefore, $rx_3(G) \leq k - 1 + \sum_{i=1}^{k} rx_3(H_i)$.

To subdivide an edge $e$ is to delete $e$, add a new vertex $x$, and join $x$ to the ends of $e$. Any graph derived from a graph $G$ by a sequence of edge subdivisions is called a *subdivision* of $G$. Given a rainbow coloring of $G$, if we subdivide an edge $e = uv$ of $G$ by $xu$ and $xv$, then we can assign $xu$ the same color as $e$ and assign $xv$ a new color, which also make the subdivision of $G$ 3-rainbow connected. Hence, the following lemma holds.

**Lemma 4.2.** Let $G$ be a connected graph, and $H$ be a subdivision of $G$. Then $rx_3(H) \leq rx_3(G) + |V(H)| - |V(G)|$.

The *Θ-graph* is a graph $G$ consisting of three internally disjoint paths with common end vertices and of lengths $a$, $b$, and $c$, respectively, such that $a \leq b \leq c$. Clearly, $a + b + c = n + 1$ where $n$ is the order of $G$.

**Lemma 4.3.** Let $G$ be a Θ-graph of order $n$. If $n \geq 7$, then $rx_3(G) \leq n - 3$.

**Proof.** Let the three internally disjoint paths be $P_1, P_2, P_3$ with the common end vertices $u$ and $v$, and the lengths of $P_1, P_2, P_3$ are $a, b, c$, respectively, where $a \leq b \leq c$.

*Case 1.* $b \geq 3$. Then $c \geq b \geq 3$, $a \geq 1$. First, we consider the graph $Θ_1$ with $a = 1$, $b = 3$ and $c = 3$. We color $uP_1v$ with color 3, $uP_2v$ with colors 2, 3, 1, and $uP_3v$ with colors 1, 3, 2. The resulting coloring makes $Θ_1$ rainbow connected. Thus, $rx_3(Θ_1) \leq 3 = |V(Θ_1)| - 3$. For a general Θ-graph $G$ with $b \geq 3$ and $n \geq 7$, we first observe that it is a subdivision of $Θ_1$. Hence by Lemma 4.2, $rx_3(G) \leq rx_3(Θ_1) + |V(G)| - |V(Θ_1)| \leq |V(G)| - 3$.

*Case 2.* $a = 1, b = 2$. Then since $a + b + c = n + 1 \geq 8$, $c \geq 5$. Consider the graph $Θ_2$ with $a = 1$, $b = 2$ and $c = 5$. We rainbow color $uP_1v$ with color 4, $uP_2v$ with colors 1, 3, and $uP_3v$ with colors 2, 3, 4, 2, 1. Thus, $rx_3(Θ_2) \leq 4 = |V(Θ_2)| -
3. Consider now a general Θ-graph $G$ with $a = 1$, $b = 2$, $c \geq 5$. Clearly, it is a subdivision of $\Theta_2$, hence by Lemma 4.2, $rx_3(G) \leq rx_3(\Theta_2) + \lvert V(G) \rvert - \lvert V(\Theta_2) \rvert \leq \lvert V(G) \rvert - 3$.

Case 3. $a = 2$, $b = 2$. Then since $a + b + c = n + 1 \geq 8$, $c \geq 4$. Consider the graph $\Theta_3$ with $a = 2$, $b = 2$ and $c = 3$. We rainbow color $uP_3v$ with colors $3, 2$, $uP_2v$ with colors $2, 1$, and $uP_3v$ with colors $1, 2, 3$. Thus, $rx_3(\Theta_3) \leq 3 = \lvert V(\Theta_3) \rvert - 3$. Consider now a general Θ-graph $G$ with $a = 2$, $b = 2$, $c \geq 4$. It is a subdivision of $\Theta_3$, hence by Lemma 4.2, $rx_3(G) \leq rx_3(\Theta_3) + \lvert V(G) \rvert - \lvert V(\Theta_3) \rvert \leq \lvert V(G) \rvert - 3$.

Every Θ-graph with $n \geq 7$ is one of the above cases, therefore $rx_3(G) \leq n - 3$. ■

A 3-sun is a graph $G$ which is defined from $C_6 = v_1, v_2, \ldots, v_6, v_7 = v_1$ by adding three edges $v_2v_4$, $v_2v_5$, and $v_3v_6$.

**Lemma 4.4.** Let $G$ be a 2-connected graph of order 6. If $G$ is a spanning subgraph of a 3-sun, then $rx_3(G) = 4$. Otherwise, $rx_3(G) = 3$.

**Proof.** Since $G$ is a 2-connected graph of order 6, $G$ is a graph with a cycle $C_6 = v_1, v_2, \ldots, v_6, v_7 = v_1$ and some additional edges. If $G$ is a subgraph of a 3-sun, then every tree connecting the three vertices $\{v_1, v_3, v_5\}$ must have size at least 4, which implies that $rx_3(G) \geq 4$. On the other hand, $rx_3(G) \leq rx_3(C_6) \leq 4$. Therefore, $rx_3(G) = 4$.

If there is an edge between the two antipodal vertices of $C_6$, then by Lemma 4.3 $rx_3(G) = 3$.

If $G$ contains the edges $v_1v_3$ and $v_2v_5$, then it contains $\Theta_3$, defined in Lemma 4.3, as a spanning subgraph, thus $rx_3(G) = 3$.

If $G$ contains the edges $v_1v_5$ and $v_2v_4$, we give a rainbow 3-coloring of $G$: $c(v_1v_2) = c(v_4v_5) = 1$, $c(v_2v_3) = c(v_2v_4) = c(v_1v_5) = c(v_3v_6) = 2$, $c(v_3v_4) = c(v_1v_6) = 3$. ■

Let $H$ be a subgraph of a graph $G$. An ear of $H$ in $G$ is a nontrivial path in $G$ whose ends are in $H$ but whose internal vertices are not. A nested sequence of graphs is a sequence $\{G_0, G_1, \ldots, G_k\}$ of graphs such that $G_i \subset G_{i+1}$, for $0 \leq i < k$. An ear decomposition of a 2-connected graph $G$ is a nested sequence $\{G_0, G_1, \ldots, G_k\}$ of 2-connected subgraphs of $G$ such that: (1) $G_0$ is a cycle; (2) $G_i = G_{i-1} \cup P_i$, where $P_i$ is an ear of $G_{i-1}$ in $G$, for $1 \leq i \leq k$; (3) $G_k = G$. We call an ear decomposition nonincreasing if $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_k)$, where $\ell(P_i)$ denotes the length of $P_i$.

**Theorem 4.5.** Let $G$ be a 2-connected graph of order $n \geq 4$. Then $rx_3(G) \leq n - 2$, with equality if and only if $G = C_n$ or $G$ is a spanning subgraph of 3-sun or $G$ is a spanning subgraph of $K_5 - e$ or $G$ is a spanning subgraph of $K_4$. 
Proof. Since $G$ is 2-connected, $G$ contains a cycle. Let $C$ be the longest cycle of $G$. Then $|V(C)| \geq 4$, $rx_3(C) \leq |V(C)| - 2$. Let $H_1 = C, H_2, H_3, \ldots, H_{n-|V(C)|+1}$ be subgraphs of $G$, each is a single vertex. Then by Lemma 4.1, $rx_3(G) \leq n - |V(C)| + rx_3(H_1) \leq n - 2$.

If $G = C$, then by Theorem 1.6, $rx_3(G) = n - 2$.

If $G \neq C$, then $G$ contains a nonincreasing ear decomposition $\{G_0, G_1, \ldots, G_k\}$. Let $H_1 = C \cup P_1$. Then $H_1$ is a $\Theta$-graph. We choose $H_2, H_3, \ldots, H_{n-|V(H_1)|+1}$ as subgraphs of $G$ with a single vertex each. By Lemma 4.1, $rx_3(G) \leq n - |V(H_1)| + rx_3(H_1)$.

If $|V(H_1)| \geq 7$, then by Lemma 4.3, $rx_3(H_1) \leq |V(H_1)| - 3$, hence $rx_3(G) \leq n - 3$.

If $|V(H_1)| = 6$, we consider three cases.

Case 1. $|V(C)| = 6$. Then $\ell(P_1) = 1$. Hence $\ell(P_1) = \ell(P_2) = \cdots = \ell(P_k) = 1$, $G$ is a graph of order 6. By Lemma 4.4, $rx_3(G) = 4$ if and only if $G$ is a spanning subgraph of a 3-sun.

Case 2. $|V(C)| = 5$. Then $\ell(P_1) = 2$. Let $u$ and $v$ be the end vertices of $P_1$. If $d_C(u, v) = 1$, then we can find a cycle larger than $C$, contradicting the choice of $C$. Otherwise, $d_C(u, v) = 2$ and is the graph $\Theta_3$ defined in Lemma 4.3. Then $rx_3(H_1) = rx_3(\Theta_3) \leq 3 = |V(H_1)| - 3$, thus $rx_3(G) \leq n - 3$.

Case 3. $|V(C)| = 4$. Then $\ell(P_1) = 3$. Let $u$ and $v$ be the end vertices of $P_1$. Either $d_C(u, v) = 1$ or $d_C(u, v) = 2$, thus we can always find a cycle larger than $C$, a contradiction.

If $|V(H_1)| = 5$, there are two cases to be considered. If $|V(C)| = 5$, then $\ell(P_1) = 1$, hence $G$ is a graph of order 5. By Theorem 2.1, $rx_3(G) = 3 = n - 2$ except for $K_5$, whose 3-rainbow index is 2. If $|V(C)| = 4$, then $\ell(P_1) = 2$. Let $u$ and $v$ be the end vertices of $P_1$. Note that $d_C(u, v) = 2$. If $\ell(P_2) = 1$, then $G$ is a graph of order 5. If $\ell(P_2) \geq 2$, then let $u'$ and $v'$ be the end vertices of $P_2$. It holds $\{u', v'\} = \{u, v\}$, otherwise, we can find a cycle larger than $C$. Let $H_1' = H_1 \cup P_2$. Then $H_1'$ is a graph consisting of 4 internally disjoint paths of length 2 with common vertices $u$ and $v$. We color the edges of the four paths with colors 12, 21, 31, 13, the resulting coloring makes $H_1'$ rainbow connected, thus, $rx_3(H_1') \leq 3 = |V(H_1')| - 3$. Let $H_2', H_3', \ldots, H_{n-|V(H_1')|+1}'$ be subgraphs of $G$, each is a single vertex. Then by Lemma 4.1, $rx_3(G) \leq n - |V(H_1')| + rx_3(H_1') \leq n - 3$.

If $|V(H_1)| = 4$, then $|V(C)| = 4$, $\ell(P_1) = 1$, $G$ is a graph of order 4, by Theorem 2.1, $rx_3(G) = 2 = n - 2$.

Therefore, $rx_3(G) = n - 2$ if and only if $G = C_n$ or $G$ is a spanning subgraph of 3-sun or $G$ is a spanning subgraph of $K_5 - e$ or $G$ is a spanning subgraph of $K_4$.

Now we turn to 2-edge-connected graphs. We say that an ear is closed if its...
endvertices are identical, otherwise, it is open. An open or closed ear is called a handle. For a 2-edge-connected graph $G$, there is a handle-decomposition, that is a sequence $\{G_0,G_1,\ldots,G_k\}$ of graphs such that: (1) $G_0$ is a cycle; (2) $G_i = G_{i-1} \cup P_i$, where $P_i$ is a handle of $G_{i-1}$ in $G$, for $1 \leq i \leq k$; (3) $G_k = G$.

Similar to Theorem 3.2, we give an upper bound of 2-edge-connected graphs.

**Theorem 4.6.** Let $G$ be a 2-edge-connected graph of order $n \geq 4$. Then $rx_3(G) \leq n - 2$, with equality if and only if $G$ is a graph attaining the upper bound in Theorem 4.5 or a graph presented in Figure 2.

**Proof.** Let $C$ be the largest cycle of $G$. If $|V(C)| \geq 4$, then $rx_3(C) \leq |V(C)| - 2$. Otherwise, all cycles of $G$ are of length 3. Since $n \geq 4$, there are at least two triangles $C_1$ and $C_2$ with a common vertex $v$. Let $F_1 = C_1 \cup C_2$, we rainbow color $F_1$ with three colors, see the graph $F_1$ in Figure 2, thus $rx_3(F_1) \leq 3 = |V(F_1)| - 2$.

Let $H_1 = C$ or $F_1$, $H_2, H_3, \ldots, H_{n-|V(H_1)|+1}$ be subgraphs of $G$ with a single vertex each. Then by Lemma 4.1, $rx_3(G) \leq n - |V(H_1)| + rx_3(H_1) \leq n - 2$.

Now we determine the graphs that obtain the upper bound $n - 2$.

If $G = C$, then by Theorem 1.6, $rx_3(G) = n - 2$.

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![Figure 2. Graphs with $rx_3(G) = n - 2$.](image)

![Figure 3. Graphs with $rx_3(G) \leq n - 3$.](image)
If $G \neq C$, then $G$ contains a handle-decomposition $\{G_0, G_1, \ldots, G_k\}$. Let $H_1 \subseteq G, H_2, H_3, \ldots, H_{n-|V(H_1)|+1}$ be subgraphs of $G$ with a single vertex each. Then by Lemma 4.1, if we show that $rx_3(H_1) \leq |V(H_1)| - 3$, then we have $rx_3(G) \leq n - 3$.

If $|V(C)| \geq 4$ and $P_1$ is an open ear, we come back to Theorem 4.5. If $|V(C)| = 3$ and $P_1$ is an open ear, then a cycle is of length larger than $C$, a contradiction.

If $|V(C)| \geq 4$ and $P_1$ is a closed ear, then $G_1$ is a union of two cycles $C_1 = C$ and $C_2 = P_1$. If both of the cycles are of length at least 4, we rainbow color each cycle $C_i$ with $|V(C_i)| - 2$ colors, which makes $G_1$ $3$-rainbow connected. So we assume that $C_2$ is of length 3. If $C_1$ is of length 5, we rainbow color $G_1$ by 4 colors, see Figure 3(1). If $C_1$ is of length greater than 5, then it is the subdivision of the graph in the case of $|V(C_1)| = 5$. For all the above three cases, we have $rx_3(G_1) \leq |V(G_1)| - 3$. Let $H_1 = G_1$, it follows that $rx_3(G) \leq n - 3$.

So it remains the case that $|V(C_1)| = 4, |V(C_2)| = 3$, we denote this graph by $F_2$, see Figure 2. Then $F_2$ is a subdivision of $F_1$, so $rx_3(F_2) \leq 4$. On the other hand, consider $S = \{v_2, v_5, v_6\}$. Every $S$-tree has size at least 4, hence $rx_3(F_2) = 4 = |V(F_2)| - 2$. Observe that $P_2$ is a closed ear of length at most 4, then $G_2 = F_2 \cup P_2$. If $\ell(P_2) = 4$, then $G_2$ contains two cycles of length 4. If $\ell(P_2) = 3$, we rainbow colors $G_2$ with $|V(G_2)| - 3$ colors, see Figure 3(2–5). For the above two cases, $rx_3(G_2) \leq |V(G_2)| - 3$. Let $H_1 = G_2$, it implies that $rx_3(G) \leq n - 3$. If $\ell(P_2) = 1$, then $P_2$ must be an edge joining the vertices of $C_1$, there are two graphs, denoted by $F_3$ and $F_4$. Similarly to $F_2$, we have $rx_3(F_3) = |V(F_3)| - 2$. For $F_4$, $rx_3(F_4) \leq rx_3(F_2) \leq 4$. On the other hand, suppose $rx_3(F_4) \leq 3$. Consider $\{v_1, v_3, v_5\}, \{v_1, v_3, v_6\}$. We have that $c(v_4v_6) = c(v_4v_5)$, which implies that there is no rainbow $\{v_1, v_5, v_6\}$-tree or $\{v_3, v_5, v_6\}$-tree, a contradiction. Hence $rx_3(F_4) = 4 = |V(F_4)| - 2$. Observe that $P_3$ is of length 1, $G_3 = F_3 \cup P_3$ or $F_4 \cup P_3$, we can rainbow color $G_3$ by 3 colors, see Figure 3(6). Let $H_1 = G_3$. Then $rx_3(G) \leq n - 3$.

If $|V(C)| = 3$ and $P_1$ is a closed ear, then $\ell(P_1) = 3$. Thus $G_1 = F_1$, and it is easy to get $rx_3(G_1) = |V(G_1)| - 2$. If $P_2$ exists, then it must be a closed ear of length 3, and there are two cases for the graph $G_2$. If $G_2$ is as in Figure 3(7), then $rx_3(G_2) \leq |V(G_2)| - 3$, let $H_1 = G_2$, thus $rx_3(G) \leq n - 3$. If $G_2$ is the graph $F_3$ in Figure 2, then we prove that its 3-rainbow index is $|V(G_2)| - 2$. Using the graph $F_3$ in Figure 2, we have that $rx_3(G_2) \leq 5$. If $rx_3(G_2) \leq 4$, then let $c : E(G) \rightarrow \{1, 2, 3, 4\}$ be the 4-rainbow coloring of $G_2$. Consider $\{v_1, v_4, v_6\}$ and $\{v_1, v_4, v_7\}$, we have $c(v_1v_3) \neq c(v_1v_6), c(v_1v_3) \neq c(v_1v_7)$. If $c(v_5v_6) = c(v_5v_7)$, then suppose that $c(v_5v_6) = 1, c(v_1v_3) = 2$. Consider $\{v_1, v_6, v_7\}$, we may assume $c(v_3v_5) = 3, c(v_6v_7) = 4$. Consider $\{v_2, v_6, v_7\}, \{v_1, v_2, v_6\}, \{v_1, v_2, v_4\}, \{v_1, v_4, v_6\}$, we have $c(v_1v_3) = 2, c(v_1v_2) = 4, c(v_3v_4) \in \{1, 4\}, c(v_4v_5) \in \{1, 4\}$, but then there is no rainbow tree connecting $\{v_4, v_6, v_7\}$. If $c(v_5v_6) \neq c(v_5v_7)$, then $c(v_1v_3) \neq c(v_2v_3)$. Let $c(v_1v_3) = 1, c(v_2v_3) = 2, c(v_5v_6) = 3, c(v_5v_6) = 4$. Consider $\{v_1, v_4, v_6\}$,
then the colors 2 and 4 must appear in the triangle $v_3v_4v_5$. Consider $\{v_2, v_4, v_7\}$, then the colors 1 and 3 must appear in the triangle $v_3v_4v_5$, which is impossible. So we consider $P_3$ and, if it exists, then it must be a close ear. There are two cases, no matter which case occurs, we can give a rainbow coloring with $|V(G_3)| - 3$ colors, see Figure 3(8–9). Let $H_1 = G_3$. Then $rx_3(G) \leq n - 3$.

Combining all the above cases, $rx_3(G) = n - 2$ if and only if $G$ is a graph attaining the upper bound in Theorem 4.5 or a graph in Figure 2.

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Received 27 December 2013
Accepted 12 February 2014