ON A SPANNING $k$-TREE IN WHICH SPECIFIED VERTICES HAVE DEGREE LESS THAN $k$

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Abstract

A $k$-tree is a tree with maximum degree at most $k$. In this paper, we give a degree sum condition for a graph to have a spanning $k$-tree in which specified vertices have degree less than $k$. We denote by $\sigma_k(G)$ the minimum value of the degree sum of $k$ independent vertices in a graph $G$. Let $k \geq 3$ and $s \geq 0$ be integers, and suppose $G$ is a connected graph and $\sigma_k(G) \geq |V(G)| + s - 1$. Then for any $s$ specified vertices, $G$ contains a spanning $k$-tree in which every specified vertex has degree less than $k$. The degree condition is sharp.

Keywords: spanning tree, degree bounded tree, degree sum condition.

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1. Introduction

All graphs considered in this paper are simple and finite. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $x$ of $G$, we denote by $\deg_G(x)$ the degree of $x$ in $G$ and by $N_G(x)$ the set of vertices adjacent to $x$ in $G$. We denote $N_G[x] = N_G(x) \cup \{x\}$, and $V_i(G)$ denotes the set of vertices of $G$ which have degree $i$ in $G$. For a subset $S$ of $V(G)$, $N_G(S) = \bigcup_{x \in S} N_G(x)$. $\alpha(G)$ denotes the independence number of $G$ and we define

$$\sigma_k(G) = \min \left\{ \sum_{x \in S} \deg_G(x_i) : S \text{ is an independent set of } G \text{ with } |S| = k \right\}$$

for $1 \leq k \leq \alpha(G)$, and $\sigma_k(G) = \infty$ if $\alpha(G) < k$.

The following is a well-known theorem on Hamiltonian cycles and paths by Ore.
Theorem 1 (Ore [6, 7]). Let \( s \) be an integer with \( 0 \leq s \leq 2 \). Suppose \( G \) is a graph with \( |V(G)| \geq 3 \) and \( \sigma_2(G) \geq |V(G)| + s - 1 \). Then the following hold:

1. if \( s = 0 \), then \( G \) has a Hamiltonian path,
2. if \( s = 1 \), then \( G \) has a Hamiltonian cycle, and
3. if \( s = 2 \), then \( G \) has a Hamiltonian path connecting any two vertices of \( G \).

We can consider a Hamiltonian path as a spanning tree with maximum degree 2.

For an integer \( k \geq 2 \), a tree \( T \) is called a \( k \)-tree if \( \deg_T(x) \leq k \) for any \( x \in V(T) \).

As we mention above, a spanning 2-tree is a Hamiltonian path.

In 1975, Win gave a degree sum condition which ensures the existence of a spanning \( k \)-tree.

Theorem 2 (Win [8]). Let \( k \geq 2 \) be an integer and \( G \) be a connected graph. If \( \sigma_k(G) \geq |V(G)| - 1 \), then \( G \) has a spanning \( k \)-tree.

Note that Theorem 2 implies Theorem 1 (1) for \( k = 2 \).

Hereafter, we consider a spanning \( k \)-tree in which every specified vertex has degree less than \( k \). Our main result is the following.

Theorem 4. Let \( k \geq 3 \) and \( s \geq 0 \) be integers, and \( G \) be a connected graph. If \( \sigma_k(G) \geq |V(G)| + s - 1 \), then for any \( s \) distinct vertices of \( G \), \( G \) has a spanning \( k \)-tree such that each specified vertex has degree less than \( k \).

We note that this is also a generalization of Theorem 2 for \( k \geq 3 \). For \( k = 2 \), we have to restrict ourselves to \( 0 \leq s \leq k = 2 \) because a spanning 2-tree has just two vertices of degree one. Then we can easily derive the same conclusion by Theorem 1.

Consider a complete bipartite graph \( G \) with parts \( X \) and \( Y \) such that \( |X| = s \) and \( |Y| = (k - 2)s + 2 \) and let \( S = X \). Then \( \sigma_k(G) = |V(G)| + s - 1 \). Suppose \( G \) has a spanning \( k \)-tree \( T \) with \( \deg_T(v) < k \) for every \( v \in S \). Then \( |V(G)| - 1 = |E(T)| \leq (k - 1)s < |V(G)| - 1 \), a contradiction. Hence \( G \) has no such a tree and the degree sum condition in Theorem 4 is sharp.
An outdirected tree $\tilde{T}$ is a rooted tree in which all the edges are directed away from the root. Let $V(\tilde{T})$ and $A(\tilde{T})$ be the vertex set and the arc set of $\tilde{T}$, respectively. For a subset $S$ of $V(\tilde{T})$, we denote by $N^+_T(S)$ the set of vertices $w$ of $V(\tilde{T})$ for which there is an arc $uw \in A(\tilde{T})$ for some $u \in S$. For a tree $T$ and $u, v \in V(T)$, let $P_T(u, v)$ be the unique path in $T$ connecting $u$ and $v$.

2. Proof of Theorem 4

If $s = 0$, we have nothing to prove since $G$ has a spanning $k$-tree by Theorem 2. So we may assume that $s \geq 1$. Let $S$ be the set of $s$ specified vertices.

By Theorem 2, $G$ has a spanning $k$-tree. Choose a spanning $k$-tree $T$ of $G$ such that $|V_k(T) \cap S|$ is as small as possible. If $V_k(T) \cap S = \emptyset$, then $T$ is a desired tree. Hence we may assume that $V_k(T) \cap S$ is not empty and let $v$ be a vertex of $S$ which have degree $k$ in $T$.

Let $T_1, \ldots, T_k$ be the connected components of $T - \{v\}$. For each $1 \leq i \leq k$, let $t_i$ be the vertex of $T_i$ which is adjacent to $v$ in $T$ and let $u_i$ be a vertex of $T_i$ with $\deg_T(u_i) = 1$.

If $u_i$ and $u_j$ are adjacent in $G$ for some $1 \leq i < j \leq k$, then $T' = T + u_iu_j - vt_i$ is a spanning $k$-tree of $G$ with $|V_k(T') \cap S| < |V_k(T) \cap S|$, a contradiction. Hence $\{u_1, \ldots, u_k\}$ is an independent set of $G$.

Let $W_1 = \bigcup_{i=2}^{k} N_G(u_i) \cap V(T_i)$.

Claim 1. $t_1$ is not contained in $W_1$.

Proof. If $t_1$ is contained in $W_1$, then $t_1$ is adjacent to $u_i$ for some $2 \leq i \leq k$. If we take $T' = T - vt_1 + t_1u_i$, then $|V_k(T') \cap S| < |V_k(T) \cap S|$, a contradiction. □

Claim 2. For each $w \in W_1$, the following statements hold.

(1) Either $\deg_T(w) = k$, or $w \in S$ and $\deg_T(w) = k - 1$.

(2) $N_G[u_1] \cap (N_T(w) \setminus V(P_T(w, u_1))) = \emptyset$.

Proof. (1) Suppose $\deg_T(w) < k$ for some $w \in W_1$. Since $w$ is adjacent to $u_i$ for some $2 \leq i \leq k$, $T' = T - tu_i + u_iw$ is also a spanning $k$-tree with $\deg_T(w) = k - 1$. If $w \notin S$, then $|V(T') \cap S| < |V(T) \cap S|$, a contradiction. If $w \in S$ and $\deg_T(w) \leq k - 2$, then also $|V(T') \cap S| < |V(T) \cap S|$. This contradicts the choice of $T$.

(2) Suppose there exists $z \in N_T(w) \setminus V(P_T(w, u_1))$ which is adjacent to $u_1$ in $G$ for some $w \in W_1$. Since $w$ is adjacent to $u_i$ in $G$ for some $2 \leq i \leq k$, $T' = T - wz - vt_1 + u_1z + wu_i$ is a spanning $k$-tree with $|V_k(T') \cap S| < |V_k(T) \cap S|$, which contradicts the choice of $T$. □

Let $W_{1,a} = \{w \in W_1 : w \notin S\}$ and $W_{1,b} = \{w \in W_1 : w \in S\}$. 


Claim 3. \(|N_T(W_1) \setminus N_G[u_1]| \geq (k - 1)|W_{1, a}| + (k - 2)|W_{1, b}|.

Proof. We may assume that \(W_1\) is not empty since otherwise the above inequality obviously holds. Furthermore, since \(t_1\) does not belong to \(W_1\) by Claim 1, \(v\) is not contained in \(N_T(W_1)\).

We consider \(T_1\) as an outdirected tree with the root \(u_1\). For any \(w_0 \in W_1\) and \(z \in N_{T_1}^+(w_0), z \notin N_G[u_1]\) holds by Claim 2 (2). This implies that \(N_{T_1}^+(w_0) \subseteq N_T(W_1) \setminus N_G[u_1]\) for every \(w_0 \in W_1\). Moreover, for any two distinct vertices \(w_1\) and \(w_2\) of \(W_1, N_{T_1}^+(w_1)\) and \(N_{T_1}^+(w_2)\) are disjoint. Consequently,

\[
|N_T(W_1) \setminus N_G[u_1]| \geq \left| \sum_{w \in W_1} N_{T_1}^+(w) \right| = (k - 1)|W_{1, a}| + (k - 2)|W_{1, b}|.
\]

\(\square\)

Claim 4. \(\sum_{i=1}^k |V(T_i) \cap N_G(u_i)| \leq |V(T_1)| - 1 + |W_{1, b}|.

Proof. By Claim 3, we obtain

\[
|V(T_1) \cap N_G(u_1)| \leq |V(T_1)| - 1 - |N_T(W_1) \setminus N_G[u_1]| \\
\leq |V(T_1)| - 1 - (k - 1)|W_{1, a}| - (k - 2)|W_{1, b}|.
\]

By the definition of \(W_1\), we have \(\sum_{i=2}^k |V(T_i) \cap N_G(u_i)| \leq (k - 1)|W_1|\). Then

\[
\sum_{i=1}^k |V(T_i) \cap N_G(u_i)| \leq |V(T_1)| - 1 + |W_{1, b}|.
\]

\(\square\)

Similarly, for each \(T_j\) we can define \(W_j, W_{j, a}, W_{j, b}\) for \(2 \leq j \leq k\). As Claim 4 we have

\[
\sum_{i=1}^k |V(T_j) \cap N_G(u_i)| \leq |V(T_j)| - 1 + |W_{j, b}|.
\]

Since \(\deg_G(u_i) \leq |\{v\}| + \sum_{j=1}^k |V(T_j) \cap N_G(u_i)|\) and \(\sum_{j=1}^k |W_{j, b}| \leq s - 1\),

\[
\sum_{i=1}^k d_G(u_i) \leq k + \sum_{i=1}^k \sum_{j=1}^k |V(T_j) \cap N_G(u_i)| \\
\leq k + \sum_{j=1}^k (|V(T_j)| - 1 + |W_{j, b}|) \\
\leq k + |V(G)| - 1 - k + s - 1 \\
= |V(G)| + s - 2,
\]

a contradiction. This completes the proof of Theorem 4.
3. Remarks

For a graph $G$, let $f$ be a mapping from $V(G)$ to positive integers and let $f^{-1}(a) = \{x \in V(G) : f(x) = a\}$ for a positive integer $a$. We call a tree $T$ to be a $f$-tree if $\deg_T(v) \leq f(v)$ for every vertex $v$ of $T$. The following sufficient conditions are already known for a graph to have a spanning $f$-tree.

**Theorem 5** (Ellingham et al. [1]). Let $G$ be a connected graph and let $f$ be a mapping from $V(G)$ to positive integers. If $w(G - S) \leq \sum_{x \in S} (f(x) - 2) + 2$, for all $S \subset V(G)$, then $G$ has a spanning $f$-tree, where $w(G - S)$ denotes the number of components of $G - S$.

**Theorem 6** (Enomoto and Ozeki [2]). Let $G$ be an $n$-connected graph and $f$ be a mapping from $V(G)$ to positive integers. Suppose $|f^{-1}(1)| + |f^{-1}(2)| \leq n + 1$ and
\[
\alpha(G) \leq \min_{R} \left\{ \sum_{x \in R} (f(x) - 1) : R \subset V(G), |R| = n \right\} + 1.
\]
Then $G$ has a spanning $f$-tree.

The above theorems are generalizations of the following classical results on spanning $k$-trees.

**Theorem 7** (Win [9]). Let $k \geq 3$ be an integer and $G$ be a connected graph. If $w(G - S) \leq (k - 2)|S| + 2$, for all $S \subset V(G)$, then $G$ has a spanning $k$-tree.

**Theorem 8** (Neumann-Lara and Rivera-Campo [5]). Let $k \geq 2$ and $n \geq 2$ be integers and $G$ be an $n$-connected graph. If $\alpha(G) \leq (k - 1)n + 1$, then $G$ has a spanning $k$-tree.

It is natural to consider a degree sum condition for a spanning $f$-tree. We pose the following conjecture.

**Conjecture 9.** Let $G$ be an $n$-connected graph, $f$ be a mapping from $V(G)$ to positive integers and let $k = \max\{f(x) : x \in V(G)\}$. Suppose $|f^{-1}(1)| \leq n$ and
\[
\sigma_k(G) \geq |V(G)| + \sum_{x \in V(G)} (k - f(x)) + 1.
\]
Then $G$ has a spanning $f$-tree.

We note that Theorems 3 and 4 partially confirm this conjecture.

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