A NOTE ON PM-COMPACT BIPARTITE GRAPHS

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Abstract

A graph is called perfect matching compact (briefly, PM-compact), if its perfect matching graph is complete. Matching-covered PM-compact bipartite graphs have been characterized. In this paper, we show that any PM-compact bipartite graph $G$ with $\delta(G) \geq 2$ has an ear decomposition such that each graph in the decomposition sequence is also PM-compact, which implies that $G$ is matching-covered.

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1. Introduction

In this paper, graphs under consideration are loopless, undirected, finite and connected. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A subset $M$ of $E(G)$ is called a perfect matching of $G$ if no two edges in $M$ are adjacent and $M$ covers all vertices of $G$. The perfect matching graph of $G$, denoted by $PM(G)$, is the graph in which each perfect matching of $G$ is a vertex and two vertices $M_1$ and $M_2$ are adjacent in $PM(G)$ if and only if the symmetric difference of $M_1$ and

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*M*₂ is an alternating cycle. The perfect matching polytope of *G* is the convex hull of the incidence vectors of all perfect matchings of *G*. Chvátal [4] shows that two vertices of the perfect matching polytope are adjacent if and only if the symmetric difference of the two perfect matchings is a cycle. This implies that *PM*(*G*) is the 1-skeleton graph of the perfect matching polytope of *G*. Naddef and Pulleyblank [5] show that if *PM*(*G*) is bipartite then *PM*(*G*) is a hypercube and otherwise *PM*(*G*) is Hamilton-connected. Bian and Zhang [1] give a sharp upper bound of the number of edges for the graphs whose perfect matching graphs are bipartite.

Padberg and Rao [6] show that, for *n* ≥ 4, the diameter of *PM*(*K₂ⁿ*) is 2 and, for *n* ∈ {2, 3}, the diameter of *PM*(*K₂ⁿ*) is 1.

Let *G* be a graph which has perfect matchings. If *PM*(*G*) is a complete graph, i.e., the diameter of the 1-skeleton graph of the perfect matching polytope of *G* is 1, we call *G* perfect matching compact, or *PM*-compact for short. Clearly, *K*₄ and *K*₆ are *PM*-compact. Let *v* be a vertex of degree 2 of *G* which has two distinct neighbors. The bicontraction of *v* is the graph obtained from *G* by contracting both edges incident with *v*. The retract of *G* is the graph obtained from *G* by successively bicontracting vertices of degree 2 until either there are no vertices of degree 2 or at most two vertices remain. A graph with two vertices and at least two parallel edges is denoted by *K*₂⁺. A graph is matching-covered if every edge of it appears in a perfect matching. Let δ(*G*) denote the minimum degree of *G*. For bipartite graphs, the following result is obtained in [7].

**Theorem 1.** (i) Let *G* be a matching-covered bipartite graph. Then *G* is *PM*-compact if and only if the retract of *G* is *K*₃,₃ or *K*₂⁺.

(ii) The graph *K*₃,₃ is the only simple matching-covered *PM*-compact bipartite graph *G* with δ(*G*) ≥ 3.

Let *H* be a subgraph of a graph *G*. An ear of *G* with respect to *H* is a path of odd length in *G* which has both ends, but no edges or interior vertices, in *H*. We call an ear trivial if it is an edge. An ear decomposition of a bipartite graph *G* is a sequence of subgraphs (*G*₀, *G*₁, . . . , *G*ᵣ), where *G*₀ = *K*₂, *G*₁ = *G*, and for 1 ≤ *i* ≤ *r*, *G*ᵢ is the union of *G*ᵢ₋₁ and an ear *P*ᵢ of *G*ᵢ with respect to *G*ᵢ₋₁. Clearly, *G*₁ is an even cycle and *G* = *K*₂ + *P*₁ + ⋯ + *P*ᵣ. In [3] Theorem 4.1.1 and Theorem 4.1.6 imply the following.

**Theorem 2.** A bipartite graph *G* is matching-covered if and only if *G* has an ear decomposition.

This theorem implies that for an ear decomposition of a matching-covered bipartite graph, each member of the sequence is matching-covered. If *G* is a matching-covered graph, then *G* is 2-connected, and so has minimum degree at least 2. In this paper, we show that a *PM*-compact bipartite graph *G* with δ(*G*) ≥ 2 has an
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ear decomposition such that each member of the decomposition sequence is \(PM\)-compact, which implies that \(G\) is matching-covered. Thus the characterization of \(PM\)-compact bipartite graphs is complete. (Note that each pendant edge (of which one end has degree 1) of a graph is contained in all perfect matchings. Using the obtained results, it is easy to characterize \(PM\)-compact bipartite graphs with minimum degree one.)

2. Main Result

A vertex \(v\) of a graph \(G\) is said to be pendant if its degree is 1 in \(G\). A bipartite graph \(G\) with bipartition \((X, Y)\) is denoted by \(G[X, Y]\). The following lemma is an immediate consequence of Exercise 16.1.13 in [2].

**Lemma 3.** Let \(G[X, Y]\) be a bipartite graph. Then \(G\) has a unique perfect matching if and only if

(i) each of \(X\) and \(Y\) contains a pendant vertex, and

(ii) when the pendant vertices and their neighbors are deleted, the resulting graph (if nonempty) has a unique perfect matching.

**Lemma 4.** Let \(G\) be a \(PM\)-compact graph and \(H\) a subgraph of \(G\) which has a perfect matching. If either (i) \(H\) is a spanning subgraph of \(G\) or (ii) \(G - V(H)\) has a perfect matching, then \(H\) is \(PM\)-compact.

**Proof.** If (i) holds, the assertion follows directly from the definition of \(PM\)-compact graphs.

If (ii) holds, let \(M\) be a perfect matching of \(G - V(H)\). Suppose that \(M_1'\) and \(M_2'\) are two distinct perfect matchings of \(H\). Then \(M_1 = M_1' \cup M\) and \(M_2 = M_2' \cup M\) are two perfect matchings of \(G\). Since \(G\) is \(PM\)-compact, \(M_1 \Delta M_2\) is an alternating cycle of \(G\). So \(M_1' \Delta M_2 = M_1 \Delta M_2\) is an alternating cycle of \(H\), and hence \(H\) is \(PM\)-compact.

**Theorem 5.** Let \(G\) be a \(PM\)-compact bipartite graph with \(\delta(G) \geq 2\). Then \(G\) has an ear decomposition \((G_0, G_1, \ldots, G_r)\) such that each \(G_i\), \(1 \leq i \leq r\), is \(PM\)-compact.

**Proof.** Suppose that \(H\) is a subgraph of \(G\) such that \(G - V(H)\) has a unique perfect matching \(M^*\). If a nontrivial ear \(P\) of \(G\) with respect to \(H\) is an \(M^*\)-alternating path, then we call \(P\) a normal ear.

**Claim.** The graph \(G\) has a normal ear with respect to \(H\).

**Proof.** To show this, write \(G^* = G - V(H)\). Let \(P^*\) be a longest \(M^*\)-alternating path in \(G^*\). Let \(x\) and \(y\) be the two ends of \(P^*\). We assert that both \(x\) and \(y\)
are covered by \( M^* \cap E(P^*) \) and each have a unique neighbor in \( G^* \), that is, their other neighbors are all in \( H \). We show this by way of contradiction. If \( x \) is not covered by \( M^* \cap E(P^*) \), let \( y' \) be the vertex matched to \( x \) under \( M^* \) (clearly, \( y' \in V(G^*) \)); otherwise, let \( y' \) be an arbitrary neighbor of \( x \) in \( G^* - E(P^*) \). When \( y' \notin V(P^*) \), \( P^* + xy' \) is an \( M^* \)-alternating path which is longer than \( P^* \). But this contradicts the choice of \( P^* \). When \( y' \in V(P^*) \), let \( C^* \) be the union of the edge \( xy' \) and the segment of \( P^* \) from \( x \) to \( y' \). Since \( G \) is bipartite, \( C^* \) is an even cycle which is an \( M^* \)-alternating cycle. Hence \( M^* \triangle E(C^*) \) is another perfect matching of \( G^* \), which contradicts the uniqueness of \( M^* \). Therefore \( x \) is covered by \( M^* \cap E(P^*) \) and has only one neighbor in \( G^* \) (namely, a member of \( V(P^*) \)). By symmetry, \( y \) also has these properties. The assertion follows.

Since \( \delta(G) \geq 2 \), by the above assertion, \( x \) and \( y \) have neighbors in \( H \). Let \( x_1, y_1 \in V(H) \) be two neighbors of \( x \) and \( y \), respectively. The above assertion also implies that the length of \( P^* \) is odd. Since \( G \) is bipartite, we have \( x_1 \neq y_1 \). Write \( P = P^* + xy_1 + yx_1 \). By the above assertion again, \( P \) is an \( M^* \)-alternating path with odd length. So \( P \) is a normal ear of \( G \) with respect to \( H \). The claim follows.

We now proceed inductively to get an ear decomposition of \( G \). For an even cycle \( C \) of \( G \), if \( G - V(C) \) has a perfect matching, we call \( C \) a \( PM \)-alternating cycle.

Recall \( \delta(G) \geq 2 \). By Lemma 3, \( G \) has at least two perfect matchings. Since each cycle in the symmetric difference of any two perfect matchings of \( G \) is a \( PM \)-alternating cycle of \( G \), \( G \) has \( PM \)-alternating cycles. Let \( C \) be a \( PM \)-alternating cycle of \( G \), and set \( H_1 = C \). If \( G - V(H_1) \) has two perfect matchings \( M^*_1 \) and \( M^*_2 \), let \( E_1 \) and \( E_2 \) be the two disjoint perfect matchings in \( H_1 \). Then \( M_1 = M^*_1 \cup E_1 \) and \( M_2 = M^*_2 \cup E_2 \) are two perfect matchings of \( G \). Since \( M_1 \triangle M_2 \) contains at least two alternating cycles, namely, \( C \) and an alternating cycle in \( M^*_1 \triangle M^*_2 \), \( M_1 \) and \( M_2 \) are not adjacent in \( PM(G) \). This contradicts the assumption that \( G \) is \( PM \)-compact. So either \( G - V(H_1) \) has a unique perfect matching, say \( M' \), or \( G - V(H_1) \) is null.

For the former case, by the above claim, \( G \) has a normal ear \( P_2 \) with respect to \( H_1 \). Set \( H_2 = H_1 + P_2 \). If \( H_2 \) is not spanning, then \( M' \setminus E(P_2) \) is the unique perfect matching of \( G - V(H_2) \). So we can proceed to find a normal ear \( P_3 \) of \( G \) with respect to \( H_2 \). Continue in this way until \( H_k = H_{k-1} + P_k, k \geq 1 \), is a spanning subgraph of \( G \). Write \( E' = E(G) \setminus E(H_k) \). Then each edge in \( E' \) is a trivial ear of \( G \) with respect to \( H_k \). Write \( r = k + |E'| \). Then we get an ear decomposition \((H_1, H_2, \ldots, H_k, \ldots, H_r)\) of \( G \), where \( H_k = H_{k-1} + P_i \) such that \( P_i \) is a normal ear of \( H_i \) with respect to \( H_{i-1} \) for each \( 2 \leq i \leq k \) and a trivial ear (an edge in \( E' \)) of \( H_i \) with respect to \( H_{i-1} \) for each \( k + 1 \leq i \leq r \).

For the latter case, \( H_1 \) is a spanning subgraph of \( G \). Then each edge in \( E' = E(G) \setminus E(H_1) \) is a trivial ear of \( G \) with respect to \( C \). Since \( G = H_1 + E' \), we are done.
Let \((G_0, G_1, \ldots, G_r)\) be an arbitrary ear decomposition of \(G\). Recall that \(G_0\) is \(K_2\) and \(G_1\) is an even cycle. To complete the proof, we show that for each \(1 \leq i \leq r-1\), \(G_i\) is \(PM\)-compact. Note that \(G - V(G_i)\) either is null or has a perfect matching (which is unique). Thus either \(G_i\) is a spanning subgraph of \(G\) or \(G - V(G_i)\) has a unique perfect matching. Since \(G_i\) also has a perfect matching, by Lemma 4, \(G_i\) is \(PM\)-compact.

Note that in the proof of Theorem 5, we show a stronger assertion that for each ear decomposition of a \(PM\)-compact bipartite graph \(G\) with \(\delta(G) \geq 2\), each member in the decomposition sequence is \(PM\)-compact.

By Theorem 2 and Theorem 5, we get the following.

**Corollary 6.** Any \(PM\)-compact bipartite graph \(G\) with \(\delta(G) \geq 2\) is matching-covered.

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### References


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