PACKING TREES INTO $n$-CHROMATIC GRAPHS

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Abstract

We show that if a sequence of trees $T_1,T_2,\ldots,T_{n-1}$ can be packed into $K_n$ then they can be also packed into any $n$-chromatic graph.

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Let $T_i$ denote a tree with $i$ edges. The author’s tree packing conjecture [3] states that $K_n$ has an edge disjoint decomposition into any given sequence $T_1,T_2,\ldots,T_{n-1}$ of trees. Gerbner, Keszegh and Palmer extended the conjecture by replacing $K_n$ with an arbitrary $n$-chromatic graph (Conjecture 2 in [1]). Here we show that the extended conjecture follows from the original one. We need the following, perhaps folklore result (Theorem 1 in [4]) and for convenience we include its simple proof.

Lemma 1. Let $G$ be a $k$-chromatic graph with a proper $k$-coloring with $k$ distinct colors. Suppose $T$ is a tree on $k$ vertices and each vertex of $T$ is labeled with a different color from the same set of $k$ colors. Then $G$ contains a subtree that is label-isomorphic to $T$ (labeled exactly the same way as $T$).

Proof. The proof is by induction, the base step is trivial for $k = 1$. Let $S_t$ denote the vertices of color $t$ in a proper $k$-coloring of a $k$-chromatic graph $G$ with a $k$-element color set $C$. Select a leaf vertex $P$ in a tree $T$ labeled with the $k$ colors of $C$, suppose its label is $i$ and assume $P$ is adjacent in $T$ with vertex $Q$ labeled with $j$. Let $A$ denote the set of vertices in $S_j$ adjacent to at least one vertex of $S_i$. Observe that $A$ is nonempty, otherwise $G$ would be $(k-1)$-chromatic. The subgraph $G^* \subset G$ obtained by removing $S_i$ and $S_j - A$ from $V(G)$ is $(k-1)$-chromatic, since the removed vertices form an independent set. Also, $G^*$ is colored with the color set $C - i$. By induction, $G^*$ contains a label-isomorphic copy of the tree $T - P$, its vertex with color $j$ is in $A$, thus adjacent to a vertex in $S_i$, extending $T - P$ to a label-isomorphic copy of $T$. □
Theorem 2. Suppose that $K_n$ has an edge disjoint decomposition into a given sequence $T_1, T_2, \ldots, T_{n-1}$ of trees and $G$ is an $n$-chromatic graph. Then $G$ contains edge disjoint copies of $T_1, T_2, \ldots, T_{n-1}$.

Proof. Let $S_1, S_2, \ldots, S_n$ be a partition of $V(G)$ into independent sets where $G$ is an $n$-chromatic graph and color all vertices of $S_i$ with color $i$. By assumption the complete graph on vertex set $V = \{1, 2, \ldots, n\}$ can be decomposed into $T_1, T_2, \ldots, T_{n-1}$. Let $G_i$ be the subgraph of $G$ induced by

$$\bigcup_{j \in V(T_i)} S_j.$$ 

The graph $G_i$ is obviously $(i + 1)$-chromatic since it has $i + 1$ color classes and a proper coloring of $G_i$ with at most $i$ colors could be obviously extended to a proper coloring of $G$ with at most $n - 1$ colors, leading to a contradiction. Applying Lemma 1 to $G_i$, we find a copy $F_i$ of $T_i$ labeled exactly the same way as $T_i$ is labeled in $K_n$. Repeating this for $i = 1, 2, \ldots, n - 1$, we obtain edge disjoint copies of $F_1, F_2, \ldots, F_{n-1}$ in $G$, in fact they are not only edge disjoint but the union of their edge sets takes exactly one edge from each bipartite graph $\{[S_i, S_j] : 1 \leq i < j \leq n\}$.

Theorem 2 allows to transfer known tree-packing results from $K_n$ to $n$-chromatic graphs. In particular, since any sequence of trees containing only paths and stars are known to be packable to $K_n ([3, 5])$, we get the following, conjectured in [2].

Corollary 3. Any sequence $T_1, T_2, \ldots, T_{n-1}$ of stars and paths is packable into any $n$-chromatic graph.

References


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