TWO SHORT PROOFS ON TOTAL DOMINATION

ALLAN BICKLE

Department of Mathematics
Western Michigan University
1903 W. Michigan
Kalamazoo, MI 49008

e-mail: allan.e.bickle@wmich.edu

Abstract

A set of vertices of a graph $G$ is a total dominating set if each vertex of $G$ is adjacent to a vertex in the set. The total domination number of a graph $\gamma_t(G)$ is the minimum size of a total dominating set. We provide a short proof of the result that $\gamma_t(G) \leq \frac{2}{3}n$ for connected graphs with $n \geq 3$ and a short characterization of the extremal graphs.

Keywords: total domination.

2010 Mathematics Subject Classification: 05C69.

A set of vertices of a graph $G$ is a total dominating set if each vertex of $G$ is adjacent to a vertex in the set. (See [3] for background.) The total domination number of a graph $\gamma_t(G)$ is the minimum size of a total dominating set. The definition immediately implies that a total dominating set is a dominating set with no isolated vertices. The total domination number is defined exactly for graphs without isolated vertices.

The following basic upper bound is due to Cockayne, Dawes, and Hedetniemi [2]. We present a shorter proof.

Theorem 1. Let $G$ be a connected graph with $n \geq 3$. Then $\gamma_t(G) \leq \frac{2}{3}n$.

Proof. Let $T$ be a spanning tree of $G$ and $v$ be a leaf of $T$. Label each vertex of $T$ with its distance from $v$ mod 3. This produces three sets that partition the vertices of $G$. Then some set contains at least one third of the vertices of $G$, and the union $S$ of the other two contains at most two thirds of the vertices. Each internal vertex of $T$ is adjacent to a vertex in each of the other sets. Replace any isolated leaves in $S$ with their neighbors. Then $S$ is a total dominating set. ■
The graphs for which $\gamma_t(G) = \lfloor \frac{2}{3}n \rfloor$ have been characterized by [1]. We present a short proof for when $\gamma_t(G) = \frac{2}{3}n$. The depth of a vertex $v$ in a tree $T$ is the minimum distance between $v$ and a leaf of $T$. A brush is a graph formed by starting with some graph $G$ and identifying a leaf of a copy of $P_3$ with each vertex of $G$.

**Theorem 2.** Let $G$ be a connected graph with $n \geq 3$. Then $\gamma_t(G) = \frac{2}{3}n$ exactly when $G$ is $C_3$, $C_6$, or a brush.

**Proof.** It is easily seen that the stated graphs are extremal, since in a brush each depth 1 vertex and a neighbor must be in the total dominating set. Let $\gamma_t(G) = \frac{2}{3}n$, so $n = 3k$. The result is obvious for $n = 3$. Let $n \geq 6$. Let $T$ be a spanning tree of $G$, so $\frac{2}{3}n \geq \gamma_t(T) \geq \gamma_t(G) = \frac{2}{3}n$, so $\gamma_t(T) = \frac{2}{3}n$. Note that no star except $K_{1,2}$ can be extremal since $\gamma_t(K_{1,2}) = 2 \leq \frac{2}{3}n$. Hence $T$ has a minimum total dominating set $S$ containing no leaves since any leaf could be replaced by a corresponding nonleaf distance two away if necessary.

Suppose that two leaves $v_1$ and $v_2$ of $T$ have a common neighbor $u$. If $T - v_1$ has a smaller total dominating set $S'$, then $u \in S'$, so $S'$ is also a total dominating set for $T$. Hence $\gamma_t(T - v_1) = |S|$, but this contradicts the upper bound, so some leaf of $T$ has a neighbor of degree 2.

If $T$ has leaves $v_1$ and $v_2$ with neighbors $u_1$ and $u_2$ with a common neighbor $w$, then $u_1$, $u_2$, and $w$ are contained in $S$. Then deleting $v_1$ and $u_1$ from $T$ only allows deleting $u_1$ from $S$, similarly contradicting the upper bound.

Suppose that deleting all depth 1 vertices of degree 2 and their neighbors produces a forest $F$. Then each isolated vertex and every leaf of each component of $F$ are already dominated. Then each component of $F$ has fewer than two-thirds of its vertices in $S$. Thus $T$ cannot achieve the upper bound, so $F$ does not exist. Thus $T$ is a brush.

Since $T$ was arbitrary, any spanning tree of $G$ is a brush. Adding edges between depth 2 vertices does not change $\gamma_t$. But adding any other edge produces a spanning tree that is not a brush unless $T = P_6$ and $G = C_6$.

A similar approach can be used to prove the characterization of the extremal graphs when $n = 3k + 2$, but the case $n = 3k + 1$ is more complicated.

**References**


doi:10.1002/net.3230100304

Received 1 July 2011
Revised 22 December 2011
Accepted 13 March 2012