VERTEX-DISTINGUISHING IE-TOTAL COLORINGS OF COMPLETE BIPARTITE GRAPHS $K_{m,n}(m < n)$

XIANG’EN CHEN, YUPING GAO AND BING YAO

College of Mathematics and Information Science
Northwest Normal University, Lanzhou 730070, P. R. China

Abstract

Let $G$ be a simple graph. An IE-total coloring $f$ of $G$ is a coloring of the vertices and edges of $G$ so that no two adjacent vertices receive the same color. Let $C(u)$ be the set of colors of vertex $u$ and edges incident to $u$ under $f$. For an IE-total coloring $f$ of $G$ using $k$ colors, if $C(u) \neq C(v)$ for any two different vertices $u$ and $v$ of $G$, then $f$ is called a $k$-vertex-distinguishing IE-total-coloring of $G$, or a $k$-VDIET coloring of $G$ for short. The minimum number of colors required for a VDIET coloring of $G$ is denoted by $\chi_{ie}^{vt}(G)$, and is called vertex-distinguishing IE-total chromatic number or the VDIET chromatic number of $G$ for short. VDIET colorings of complete bipartite graphs $K_{m,n}(m < n)$ are discussed in this paper. Particularly, the VDIET chromatic numbers of $K_{m,n}(1 \leq m \leq 7, m < n)$ as well as complete graphs $K_n$ are obtained.

Keywords: complete bipartite graphs, IE-total coloring, vertex-distinguishing IE-total coloring, vertex-distinguishing IE-total chromatic number.

2010 Mathematics Subject Classification: 05C15.
for a vertex-distinguishing proper edge coloring of a \( vdec \)-graph \( G \) is denoted by \( \chi'_s(G) \). Vertex-distinguishing proper edge coloring has been considered in several papers [1-5, 8-9].

A general edge coloring (not necessarily proper) of a graph \( G \) is said to be vertex-distinguishing if \( S(u) \neq S(v) \) is required for any two distinct vertices \( u, v \). The point-distinguishing chromatic index of a \( vdec \)-graph \( G \), denoted by \( \chi_0(G) \), refers to the minimum number of colors required for a vertex-distinguishing general edge coloring of \( G \). This parameter was introduced by Harary and Plantholt in [7]. Although the structure of complete bipartite graph is simple, it seems that the problem of determining \( \chi_0(K_{m,n}) \) is not easy, especially in the case \( m = n \), as documented by papers of Hornák and Soták [10, 11], Zagaglia Salvi [13, 14] and Hornák and Zagaglia Salvi [12].

A total coloring of a graph \( G \) is an assignment of some colors to the vertices and edges of \( G \). It is proper if the following three conditions are satisfied:

- **Condition (v):** No two adjacent vertices receive the same color;
- **Condition (e):** No two adjacent edges receive the same color;
- **Condition (i):** No edge receives the same color as any one of its incident vertices.

For a total coloring (proper or not) \( f \) of \( G \) and a vertex \( v \) of \( G \), denote by \( C_f(v) \), or simply \( C(v) \) if no confusion arise, the set of colors used to color the vertex \( v \) as well as the edges incident to \( v \). Let \( \overline{C}(v) \) be the complementary set of \( C(v) \) in the set of all colors we used. Obviously \( |C(v)| \leq d_G(v) + 1 \) and the equality holds if the total coloring is proper.

For a proper total coloring, if \( C(u) \neq C(v) \) for any two distinct vertices \( u \) and \( v \), then the coloring is called a vertex-distinguishing proper total coloring and the minimum number of colors required for a vertex-distinguishing proper total coloring is denoted by \( \chi_{vt}(G) \). This concept was considered in [6, 15]. In [15], the following conjecture was given.

**Conjecture 1.** Suppose \( G \) is a simple graph and \( n_d \) is the number of vertices of degree \( d \), \( \delta \leq d \leq \Delta \). Let \( k \) be the minimum positive integer such that \( \binom{k}{d+1} \geq n_d \) for all \( d \) such that \( \delta \leq d \leq \Delta \). Then \( \chi_{vt}(G) = k \) or \( k + 1 \).

From [15] we know that the above conjecture is valid for complete graphs, complete bipartite graphs, paths and cycles, etc.

In this paper we propose a kind of vertex-distinguishing general total coloring called IE-total coloring. The relationship between this coloring and vertex-distinguishing proper total coloring is similar to the relationship between vertex-distinguishing general edge coloring and vertex-distinguishing proper edge coloring.

An **IE-total coloring** of a graph \( G \) is a total coloring of \( G \) such that the Condition (v) is satisfied. If \( f \) is an IE-total coloring of graph \( G \) using \( k \) colors
and for any $u,v \in V(G)$, $u \neq v$, we have $C(u) \neq C(v)$, then $f$ is called a $k$-vertex-distinguishing IE-total coloring, or a $k$-VDIET coloring. The number

$$\min\{k : G \text{ has a } k\text{-VDIET coloring}\}$$

is called the vertex-distinguishing IE-total chromatic number of a graph $G$ and is denoted by $\chi_{ie}^{vt}(G)$.

The following proposition is obviously true.

**Proposition 2.** $\chi_{ie}^{vt}(G) \leq \chi_{vt}(G)$.

For a graph $G$, let $n_i$ denote the number of vertices of degree $i$, $\delta \leq i \leq \Delta$. Let

$$\xi(G) = \min \left\{ k \mid \binom{k}{1} \left( \binom{k}{2} + \binom{k}{3} + \cdots + \binom{k}{s} + 1 \right) \geq n_{\delta} + n_{\delta+1} + \cdots + n_s, \delta \leq s \leq \Delta \right\}.$$ 

Obviously, $\chi_{ie}^{vt}(G) \geq \xi(G)$.

In the following we will consider the VDIET colorings of complete bipartite graphs $K_{m,n}$ ($1 \leq m < n$) and complete graphs $K_n$, then we will give three conjectures.

For a complete bipartite graph $K_{m,n}$ ($1 \leq m < n$), $\xi(K_{m,n})$ is the minimum positive integer $l$ such that

(1) $$\binom{l}{1} + \binom{l}{2} + \binom{l}{3} + \cdots + \binom{l}{m+1} \geq n,$$

(2) $$\binom{l}{1} + \binom{l}{2} + \binom{l}{3} + \cdots + \binom{l}{n+1} \geq n + m.$$

**Proposition 3.**

(i) If $n = \sum_{i=1}^{m+1} \binom{m+2}{i} - m + 1$, then $\xi(K_{m,n}) = m + 2$;

(ii) If $\sum_{i=1}^{m+1} \binom{m+2}{i} - m + 2 \leq n \leq \sum_{i=1}^{m+1} \binom{m+3}{i} - m$, then $\xi(K_{m,n}) = m + 3$.

**Proof.**

(i) When $l = m + 1$, (1) is not valid, because

$$\binom{m+1}{1} + \binom{m+1}{2} + \cdots + \binom{m+1}{m+1} = 2^{m+1} - 1,$$

$$n = 2^{m+2} - 2 - m + 1 = 2^{m+2} - m - 1 > 2^{m+1} - 1.$$

Therefore $\xi(K_{m,n}) \geq m + 2$. Since

$$\binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+2}{m+1} = 2^{m+2} - 2 \geq 2^{m+2} - m - 1 = n,$$

$$\binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+2}{n+1} = 2^{m+2} - 1 = m + n.$$
so we have \( \xi(K_{m,n}) = m + 2 \).

(ii) When \( \sum_{i=1}^{m+1} \binom{m+2}{i} - m + 2 \leq n \leq \sum_{i=1}^{m+1} \binom{m+3}{i} - m \), i.e., \( 2^{m+2} - m \leq n \leq 2^{m+3} - (m + 3) - 2 - m = 2^{m+3} - 2m - 5 \), we have
\[
\binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+2}{n+1} = 2^{m+2} - 1 \leq m + n - 1.
\]
Therefore, (2) is not valid if \( l = m + 2 \). So, \( \xi(K_{m,n}) \geq m + 3 \). When \( l = m + 3 \), (1) and (2) are right, so \( \xi(K_{m,n}) = m + 3 \). ■

**Proposition 4.**

(i) If \( \sum_{i=1}^{m+1} \binom{k-1}{i} - m < n \leq \sum_{i=1}^{m+1} \binom{k-1}{i} \) and \( k \geq m + 4 \), then \( \xi(K_{m,n}) = k - 1 \);

(ii) If \( \sum_{i=1}^{m+1} \binom{k-1}{i} < n \leq \sum_{i=1}^{m+1} \binom{k-1}{i} - m \) and \( k \geq m + 4 \), then \( \xi(K_{m,n}) = k \).

**Proof.**

(i) As
\[
\binom{k-2}{1} + \binom{k-2}{2} + \cdots + \binom{k-2}{m+1} \leq \left[ \binom{k-2}{0} + \binom{k-2}{1} \right] + \left[ \binom{k-2}{1} + \binom{k-2}{2} \right] + \cdots + \left[ \binom{k-2}{m} + \binom{k-2}{m+1} \right] - m = \left( \binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{m+1} \right) - m < n,
\]
(1) is not valid if \( l = k - 2 \). Therefore, \( \xi(K_{m,n}) \geq k - 1 \). Because
\[
\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{m+1} \geq n,
\]
\[
\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{n+1} \geq n + \binom{k-1}{m+2} + \binom{k-1}{m+3} + \cdots + \binom{k-1}{n+1} > m + n,
\]
so (1) and (2) are valid if \( l = k - 1 \). We have \( \xi(K_{m,n}) = k - 1 \).

(ii) When \( \sum_{i=1}^{m+1} \binom{k-1}{i} < n \leq \sum_{i=1}^{m+1} \binom{k-1}{i} - m \), (1) is not valid if \( l = k - 1 \), whereas (1) and (2) are valid if \( l = k \). Therefore \( \xi(K_{m,n}) = k \). ■

**Theorem 5.** Let \( m \geq 1 \), \( n > 2^{m+2} - m - 2 \). Then \( \chi_{vl}(K_{m,n}) = k \) when \( \sum_{i=1}^{m+1} \binom{k-1}{i} - m < n \leq \sum_{i=1}^{m+1} \binom{k}{i} - m \).

**Proof.** As \( n > 2^{m+2} - m - 2 \), we have \( k > m + 2 \) (otherwise, if \( k \leq m + 2 \), then
\[
n \leq \sum_{i=1}^{m+1} \binom{k}{i} - m \leq \sum_{i=1}^{m+1} \binom{m+2}{i} - m = 2^{m+2} - 2m - m, \]a contradiction).
We prove that $K_{m,n}$ does not have a $(k-1)$-VDIET coloring. If not, suppose $g$ is a VDIET coloring of $K_{m,n}$ using colors $1, 2, \ldots, k-1$. Let $B_0 = \{g(u_1), g(u_2), \ldots, g(u_m)\}$, $B_i = \{1, 2, \ldots, k-1\} \setminus C_g(u_i)$, $i = 1, 2, \ldots, m$. Note that none of $B_0, B_1, B_2, \ldots, B_m$ is the color set of any vertex $v_j$. Let $T = \{j : |C_g(v_j)| = 1, j = 1, 2, \ldots, n\}$ and $t = |T|$. Then $B_0 \cap \{g(v_j)\}j \in T\} = \emptyset$. Without loss of generality, we assume that $C_g(v_j) = \{j\}$, $j = 1, 2, \ldots, t$, then we have $|C_g(v_j)| \geq 2, j = t + 1, \ldots, n$ and $C_g(u_i) \supseteq \{1, 2, \ldots, t, g(u_i)\}, i = 1, 2, \ldots, m$.

Case 1. $t \geq k - m - 3$. For each $i \in \{1, 2, \ldots, m\}$, we have $|C_g(u_i)| \geq t + 1$ and $|B_i| \leq (k-1) - (t+1) \leq (k-1) - (k-m-3+1) = m+1$. Note that $|B_0| \leq m+1$ and none of $B_0, B_1, B_2, \ldots, B_m$ is the color set of any vertex $v_j$. Therefore there are at most $(k-1) + (k-1) + \cdots + (k-1) - m$ subsets of $\{1, 2, \ldots, k-1\}$ with cardinality between 1 and $m + 1$. which may become the color sets of vertices $v_1, v_2, \ldots, v_n$. This is a contradiction.

Case 2. $t \leq k - m - 4$. In this case, there are at least $(k-1) - (k-m-4) = m + 3$ subsets of $\{1, 2, \ldots, k-1\}$ with cardinality 1 which are not the color sets of vertices $v_1, v_2, \ldots, v_n$. This is also a contradiction because $(k-1) + (k-1) + \cdots + (k-1) - (m+3) < (k-1) + (k-1) + \cdots + (k-1) - m < n$, and at most $(k-1) + (k-1) + \cdots + (k-1) - (m+3)$ subsets of $\{1, 2, \ldots, k-1\}$ with cardinality between 1 and $m + 1$ cannot distinguish $n$ vertices.

In the following we prove that $K_{m,n}$ has a $k$-VDIET coloring. Let $V(K_{m,n}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ and $E(K_{m,n}) = \{u_iv_j : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$.

Put $D(u_i) = \{1, 2, \ldots, k\} \setminus \{i\}, i = 1, 2, \ldots, m-1$, $D(u_m) = \{1, 2, \ldots, k\}$, $D(v_j) = \{j, k\}, j = 1, 2, \ldots, m-1$, $D(v_j) = \{j\}, j = m, m+1, \ldots, k-1$.

Now distribute other subsets of $\{1, 2, \ldots, k\}$ with cardinality between 2 and $m + 1$ to vertices $v_k, v_{k+1}, \ldots, v_n$. These $n - k + 1$ subsets are denoted by $D(v_k), D(v_{k+1}), \ldots, D(v_n)$, respectively.

Construct a mapping $f$ from $V(K_{m,n}) \cup E(K_{m,n})$ to $\{1, 2, \ldots, k\}$ as follows:

Put $f(u_i) = k, i = 1, 2, \ldots, m, f(v_j) = \min D(v_j), j = 1, 2, \ldots, n,$

$f(u_i, v_j) = k$ for $i = 1, 2, \ldots, m-1$, $f(u_m, v_j) = m$,

$f(u_i, v_j) = j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, k-1, i \neq j$.

For each $j = k, k+1, \ldots, n$, we recursively let $f(u_i, v_j) = \min \{D(u_i) \cap (D(v_j) \setminus \{f(v_j)\})$ or $f(u_i, v_j) \in D(u_i) \cap D(v_j)$ when $D(u_i) \cap (D(v_j) \setminus \{f(v_j)\}) = \emptyset$.

When $2 \leq i \leq m$, $f(u_i, v_j) = \min \{D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_1, v_j), f(u_2, v_j), \ldots, f(u_{i-1}, v_j)\})$ or $f(u_i, v_j) \in D(u_i) \cap D(v_j)$ when $D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_1, v_j), f(u_2, v_j), \ldots, f(u_{i-1}, v_j)\}) = \emptyset$.

It is not hard to see that $C_f(u_i) = D(u_i), i = 1, 2, \ldots, m; C_f(v_j) = D(v_j), j = 1, 2, \ldots, n$ and moreover $f(u_i) > f(v_j)$, therefore our coloring $f$ is a vertex distinguishing IE-total coloring and then $\chi^*_t(K_{m,n}) \leq k$. ■
Theorem 6. Let \( m \geq 2 \), \((m+2)\choose 1 + (m+2)\choose 2 + \cdots + (m+2)\choose m+1\) - 2m + 1 < n \leq (m+2)\choose 1 + (m+2)\choose 2 + \cdots + (m+2)\choose m+1\), i.e., \( 2m^2 - 2m - 1 < n \leq 2m^2 - m - 2 \). Then \( \chi_{vt}(K_{m,n}) = m + 3 \).

Proof. When \( 2m^2 - 2m - 1 < n \leq 2m^2 - m - 2 \), we have \( \chi_{vt}(K_{m,n}) \geq \xi(K_{m,n}) = m + 2 \). We first prove that \( K_{m,n} \) does not have a \((m + 2)\)-VDIET coloring.

Otherwise, suppose \( g \) is a VDIET coloring of \( K_{m,n} \) using colors \( 1, 2, \ldots, m + 2 \).

Let \( B_0 = \{ g(u_1), g(u_2), \ldots, g(u_m) \} \), \( B_i = \{ 1, 2, \ldots, m + 2 \} \setminus C(g(u_i), i = 1, 2, \ldots, m \). Note that \( B_0, B_1, B_2, \ldots, B_m \) are distinct and at most one of them is an empty set. \( B_0, B_1, B_2, \ldots, B_m \) are not the color sets of vertices \( v_1, v_2, \ldots, v_n \). We give a fact as follows.

Observation 7. \( |C(g(u_i))| \geq 2, i = 1, 2, \ldots, m \). Furthermore, there exists a vertex \( v \in \{ v_1, v_2, \ldots, v_n \} \) with \( |C(g(v))| = 1 \).

Proof. Suppose that there exists a vertex \( u_i \in \{ u_1, u_2, \ldots, u_m \} \) with \( C(g(u_i)) = \{ \alpha \} \), \( \alpha \in \{ 1, 2, \ldots, m + 2 \} \). Then \( \alpha \in C(g(v_j), j = 1, 2, \ldots, n \). However, \( 2m^2 + 2m - 1 < 2m^2 - 2m - 1 < n \), i.e., the subsets of \( \{ 1, 2, \ldots, m + 2 \} \) which contain \( \alpha \) cannot distinguish \( n \) vertices, this is a contradiction. Therefore, \( |C(g(u_i))| \geq 2, i = 1, 2, \ldots, m \).

Suppose \( |C(g(v_j))| \geq 2, j = 1, 2, \ldots, n \), i.e., all 1-subsets of \( \{ 1, 2, \ldots, m + 2 \} \) are not the color sets of vertices \( u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \). Therefore, there are at most \( 2m^2 - 2m - 1 - (m + 2) < 2m^2 - 2m - 1 - m < m + n \) nonempty subsets of \( \{ 1, 2, \ldots, m + 2 \} \) which may become the color sets of vertices \( u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \). This is a contradiction. \( \square \)

Using the above observation, without loss of generality, we assume \( C(g(v_1)) = \{ 1 \} \). Then \( 1 \in C(g(u_i)), i = 1, 2, \ldots, m, g(u_i) \neq 1, i = 1, 2, \ldots, m \).

It is obvious that \( B_0, B_1, B_2, \ldots, B_m \) are not the color sets of any vertex \( u_i, i = 1, 2, \ldots, m \). Therefore, there are at most \( 2m^2 - 2m - 1 - m < m + n \) nonempty subsets of \( \{ 1, 2, \ldots, m + 2 \} \) which may become the color sets of vertices \( u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \). This is a contradiction.\( \square \)

In the following we prove that \( K_{m,n} \) has a \((m + 3)\)-VDIET coloring when \( 2m^2 - 2m - 1 < n \leq 2m^2 - m - 2 \).

By Theorem 5, we can give \( K_{m,t} \) a \((m + 3)\)-VDIET coloring \( f \) using colors \( 1, 2, \ldots, m + 3 \), where \( 2m^2 - 2m - 2 - m < t \leq 2m^3 - 2m - 5 \). Now delete the vertices \( v_{n+1}, v_{n+2}, \ldots, v_t \) and their colors, delete the edges \( u_iv_j, i = 1, 2, \ldots, m, j = n + 1, n + 2, \ldots, t \) and their colors. It is not hard to see that under the resulting coloring the color sets of \( u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \) do not change, so we get a \((m + 3)\)-VDIET coloring \( g \) of \( K_{m,n} \) using colors \( 1, 2, \ldots, m + 3 \). \( \square \)
Theorem 8. Let $s$ be the minimum positive integer such that $2^s - 1 \geq 3m$. When $2^r - 2m - 1 < n \leq 2^{r+1} - 2m - 1$, we have $\chi^e_{VDIET}(K_{m,n}) = r + 1$, where $r = m + 1, m, m - 1$ and $r \geq s$.

Proof. \(\xi(K_{m,n}) = \begin{cases} r, & \text{when } 2^r - 2m - 1 < n \leq 2^{r+1} - 2m - 1; \\ r + 1, & \text{when } 2^{r-1} - 2m - 1 < n \leq 2^r - m - 1. \end{cases}\)

When $2^r - 2m - 1 < n \leq 2^{r+1} - 2m - 1$, it is obvious that $\chi^e_{VDIET}(K_{m,n}) \geq r$. We prove that $K_{m,n}$ does not have an $r$-VDIET coloring when $r = m + 1, m, m - 1$. If not, let $g$ be an $r$-VDIET coloring of $K_{m,n}$ using colors $1, 2, \ldots, r$. First we give four claims as follows.

Claim 9. \(|C(v_j)| \geq 2, j = 1, 2, \ldots, n.\)

Proof. Suppose the claim is not true, without loss of generality, we assume $C(v_1) = \{1\}$. Then $1 \in C(u_i), i = 1, 2, \ldots, m$. Let $B_0 = \{g(u_1), g(u_2), \ldots, g(u_m)\}$, $B_i = \{1, 2, \ldots, r\} \setminus C(u_i), i = 1, 2, \ldots, m$. Note that $1 \notin B_0, 1 \notin B_i, i = 1, 2, \ldots, m$. We have $B_0, B_1, B_2, \ldots, B_m$ are distinct and not the color sets of vertices $u_1, u_2, \ldots, u_m$. Moreover, none of $B_0, B_1, B_2, \ldots, B_m$ is the color set of any vertex $v_j, j = 1, 2, \ldots, n$, because $C(u_i) \cap C(v_j) = \emptyset, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. And two adjacent vertices must have different colors. At most one of $B_0, B_1, B_2, \ldots, B_m$ is an empty set, so there are at most $2^r - 1 - m$ nonempty subsets of $\{1, 2, \ldots, r\}$ which are available for the vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. However, $2^r - 1 - m < m + n$, i.e., these subsets cannot distinguish $m + n$ vertices, this is a contradiction.

Claim 10. \(|C(u_i)| \geq 2, i = 1, 2, \ldots, m.\)

Proof. Suppose the claim is not true. Without loss of generality we assume $C(u_1) = \{1\}$. Then $1 \in C(v_j), j = 1, 2, \ldots, n$. Thus, $\overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ are not available for vertices $v_1, v_2, \ldots, v_n$. Moreover, $\overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ cannot be the color sets of vertices $u_1, u_2, \ldots, u_m$ because $C(u_i) \cap C(v_j) \neq \emptyset$. At most one of $\overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n)$ is an empty set, so there are at most $2^r - 1 - (n - 1)$ nonempty subsets of $\{1, 2, \ldots, r\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. However, $2^r - 1 - (n - 1) \leq 2^r - 1 - m < m + n$, these subsets cannot distinguish $m + n$ vertices, this is a contradiction.

Claim 11. $C(u_1) \cap C(u_2) \cap \cdots \cap C(u_m) = \emptyset$.

Proof. Suppose $1 \in C(u_i), i = 1, 2, \ldots, m$. Then the $m + 1$ distinct subsets $\{1\}, \overline{C}(u_1), \overline{C}(u_2), \ldots, \overline{C}(u_m)$ are not available for any vertex, and at most one of them is an empty set. Then there are at most $2^r - 1 - m$ subsets of $\{1, 2, \ldots, r\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$. However, $2^r - 1 - m < m + n$, so these subsets cannot distinguish $m + n$ vertices, this is a contradiction.
Claim 12. \( C(v_1) \cap C(v_2) \cap \cdots \cap C(v_n) = \emptyset. \)

**Proof.** Suppose \( 1 \in C(v_j), j = 1, 2, \ldots, n. \) Then the \( n+1 \) distinct subsets \( \{1\}, \overline{C}(v_1), \overline{C}(v_2), \ldots, \overline{C}(v_n) \) are not available for any vertex, and at most one of them is an empty set. The remaining 2\(^r\) - 1 - n subsets of \( \{1, 2, \ldots, r\} \) cannot distinguish \( m+n \) vertices because 2\(^r\) - 1 - n \( \leq 2^r - 1 - m < m+n \), this is a contradiction.

Now we consider two cases.

**Case 1.** \( r = m, m+1 \). By Claims 9 and 10, all 1-subsets of \( \{1, 2, \ldots, r\} \) cannot be the color sets of any vertex. So there are at most 2\(^r\) - 1 - r \( \leq 2^r - m - 1 < m+n \) subsets of \( \{1, 2, \ldots, r\} \) which are available for vertices \( u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \). This is a contradiction.

**Case 2.** \( r = m-1 \). By Claims 9 and 10, all the 1-subsets \( \{1\}, \{2\}, \ldots, \{m-1\} \) cannot be the color sets of any vertex. The remaining 2\(^m\) - 1 - (m - 1) = 2\(^m\) - m subsets of \( \{1, 2, \ldots, m-1\} \) cannot distinguish \( m+n \) vertices when 2\(^m\) - 2m < n \( \leq 2^m - 1 - m - 1 \), this is a contradiction, so \( K_{m,n} \) does not have an \((m-1)\)-VDIET coloring when 2\(^m\) - 2m < n \( \leq 2^m - 1 - m - 1 \).

Now we consider the case n = 2\(^m\) - 2m. Let \( t = |\{g(u_1), g(u_2), \ldots, g(u_m)\}| \). Without loss of generality we assume \( \{g(u_1), g(u_2), \ldots, g(u_m)\} = \{1, 2, \ldots, t\} \). By Claims 11 and 12 we know that 2 \( \leq t \leq r-2 \), thus if \( r \leq 3 \), this is a contradiction. So \( r \geq 4 \). None of 2-subsets of \( \{1, 2, \ldots, t\} \) is available for \( v_1, v_2, \ldots, v_n \).

If \( \{1, 2\} \notin \{C(u_1), C(u_2), \ldots, C(u_m)\} \), then at most 2\(^m\) - 1 - m < m+n subsets of \( \{1, 2, \ldots, m-1\} \) are available for vertices \( u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \), this is a contradiction.

Therefore, \( \{1, 2\} \in \{C(u_1), C(u_2), \ldots, C(u_m)\} \). Without loss of generality, assume \( C(u_1) = \{1, 2\} \). By Claim 12, there are at least two colors among \( v_1, v_2, \ldots, v_n \), say \( t+1, t+2 \). Then \( \{t+1, t+2\} \notin \{C(u_1), C(u_2), \ldots, C(u_m)\} \).

If \( \{t+1, t+2\} \notin \{C(v_1), C(v_2), \ldots, C(v_n)\} \), then at most 2\(^m\) - 1 - m < m+n subsets of \( \{1, 2, \ldots, m-1\} \) are available for vertices \( u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \), this is a contradiction. Thus \( \{t+1, t+2\} \in \{C(v_1), C(v_2), \ldots, C(v_n)\} \). Then \( t+1 \in C(u_i) \) or \( t+2 \in C(u_i) \), \( i = 1, 2, \ldots, m, \) However, \( C(u_1) = \{1, 2\} \), this is a contradiction.

So, \( K_{m,n} \) does not have an \( r \)-VDIET coloring when 2\(^m\) - 2m \( \leq n \leq 2^m - 1 - m - 1 \) and \( r = m+1, m, m-1 \).

In the following we give an \((r+1)\)-VDIET coloring of \( K_{m,n} \) using colors \( 1, 2, \ldots, r, r+1 \), where \( r = m-1, m, m+1 \).

Let \( V(K_{m,n}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\} \) and \( E(K_{m,n}) = \{u_iv_j : i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\} \).

Put \( D(u_i) = \{1, 2, \ldots, r+1\} \setminus \{i\}, i = 1, 2, \ldots, m-1, D(u_m) = \{1, 2, \ldots, r+1\}; D(v_j) = \{j, r+1\}, j = 1, 2, \ldots, m-1 \).
When \( r = m + 1 \), put \( D(v_{r-1}) = \{r - 1\}, D(v_r) = \{r\} \). When \( r = m \), put \( D(v_r) = \{r\} \).

Now distribute other subsets of \( \{1, 2, \ldots, r + 1\} \) with cardinality between 2 and \( r \) to vertices \( v_{r+1}, v_{r+2}, \ldots, v_n \). These \( n - r \) subsets are denoted by \( D(v_{r+1}), D(v_{r+2}), \ldots, D(v_n) \), respectively.

Construct a mapping \( f \) from \( V(K_{m,n}) \cup E(K_{m,n}) \) to \( \{1, 2, \ldots, r+1\} \) as follows:

Put \( f(u_i) = r + 1, \) \( i = 1, 2, \ldots, m, \) \( f(v_j) = \min D(v_j), \) \( j = 1, 2, \ldots, n, \) \( f(u_i v_i) = r + 1 \) for \( i = 1, 2, \ldots, m - 1, \) \( f(u_i v_j) = j, \) \( i = 1, 2, \ldots, m, j = 2, \ldots, m - 1, \) \( i \neq j, \) \( f(u_i v_j) = j, i = 1, 2, \ldots, m, j = m, \ldots, r \) (if \( r = m \) or \( m + 1 \)).

For each \( j = r + 1, r + 2, \ldots, n \), we recursively let \( f(u_i v_j) = \min \{D(u_i) \cap (D(v_j) \setminus \{f(v_j)\})\} \) or \( f(u_i v_j) \in D(u_i) \cap D(v_j) \) when \( D(u_i) \cap D(v_j) \setminus \{f(v_j)\} \) = \( \emptyset \).

When \( 2 \leq i \leq m,\) \( f(u_i v_j) = \min \{D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_i v_j), f(u_2 v_j), \ldots, f(u_{i-1} v_j)\})\} \) or \( f(u_i v_j) \in D(u_i) \cap D(v_j) \) when \( D(u_i) \cap D(v_j) \setminus \{f(v_j), f(u_1 v_j), f(u_2 v_j), \ldots, f(u_{i-1} v_j)\} \) = \( \emptyset \).

It is not hard to see that \( C_f(u_i) = D(u_i), \) \( i = 1, 2, \ldots, m; C_f(v_j) = D(v_j), j = 1, 2, \ldots, n \) and moreover \( f(u_i) > f(v_j) \), therefore our coloring \( f \) is a vertex distinguishing IE-total coloring and then \( \chi_{\text{ie}}^\text{IE}(K_{m,n}) \leq r + 1, r = m - 1, m + 1. \)

So \( \chi_{\text{ie}}^\text{IE}(K_{m,n}) = r + 1, r = m - 1, m + 1. \)

**Theorem 13.** \( \chi_{\text{ie}}^\text{IE}(K_1,n) = \begin{cases} 
2, & \text{when } n = 1; \\
3, & \text{when } n = 2; \\
k, & \text{when } \binom{k-1}{1} + \binom{k-1}{2} - 1 < n \leq \binom{k}{1} + \binom{k}{2} - 1, k \geq 3. 
\end{cases} \)

**Proof.** It is easy to prove the theorem in the case \( n = 1, 2 \). By Theorem 5, this theorem is valid when \( \binom{k-1}{1} + \binom{k-1}{2} - 1 < n \leq \binom{k}{1} + \binom{k}{2} - 1, k \geq 3. \)

**Theorem 14.** \( \chi_{\text{ie}}^\text{IE}(K_2,n) = \begin{cases} 
3, & \text{when } n = 2, 3; \\
4, & \text{when } n = 4, 5, \ldots, 11; \\
5, & \text{when } n = 12; \\
k, & \text{when } \binom{k-1}{1} + \binom{k-1}{2} + \binom{k-1}{3} - 2 < n \leq \binom{k}{1} + \binom{k}{2} + \binom{k}{3} - 2, k \geq 5. 
\end{cases} \)

**Proof.** By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when \( n \geq 4. \) Now we consider the case \( n = 2, 3. \) It is obvious that \( \chi_{\text{ie}}^\text{IE}(K_{2,n}) \geq \xi(K_{2,n}) = 3 \) when \( n = 2, 3. \) Let \( V(K_{2,n}) = \{u_1, u_2, v_1, v_2, \ldots, v_n\} \) and \( E(K_{2,n}) = \{u_i v_j : 1 \leq i \leq 2, 1 \leq j \leq n\} \). We give a 3-VDIET coloring of \( K_{2,n} \) using colors 1, 2, 3 when \( n = 2, 3. \)

Let \( u_1, u_2 \) receive color 1, \( v_1 \) and its incident edges receive color 2. We assign color 3, 3, 1 to \( v_2, u_1 v_2, u_2 v_2, \) respectively. And when \( n = 3, \) we assign color 2, 3, 2 to \( v_3, u_1 v_3, u_2 v_3, \) respectively.
Then under the above coloring, we have \( C(u_1) = \{1, 2, 3\} \), \( C(u_2) = \{1, 2\} \), \( C(v_1) = \{2\} \), \( C(v_2) = \{1, 3\} \) and \( C(v_3) = \{2, 3\} \) (when \( n = 3 \)). Thus the above coloring is a VDIET coloring of \( K_{2,n}(n = 2, 3) \) using 3 colors. \( \square \)

\[ \text{Theorem 15.} \quad \chi_{\text{et}}(K_{3,n}) = \begin{cases} 4, & \text{when } 3 \leq n \leq 9; \\ 5, & \text{when } 10 \leq n \leq 25; \\ 6, & \text{when } n = 26, 27; \\ k, & \text{when } \left(\begin{array}{c} k-1 \\ 1 \end{array}\right) + \cdots + \left(\begin{array}{c} k-1 \\ 4 \end{array}\right) - 3 < n \leq \left(\begin{array}{c} k \\ 1 \end{array}\right) + \cdots + \left(\begin{array}{c} k \\ 4 \end{array}\right) - 3, k \geq 6. \end{cases} \]

\[ \text{Proof.} \quad \text{By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when } n \geq 10. \text{ Now we consider the case } 3 \leq n \leq 9. \]

\[ \xi(K_{3,n}) = \begin{cases} 3, & \text{when } n = 3, 4; \\ 4, & \text{when } 5 \leq n \leq 9. \end{cases} \]

Let \( V(K_{3,n}) = \{u_1, u_2, u_3, v_1, v_2, \ldots, v_n\} \) and \( E(K_{3,n}) = \{u_iv_j : 1 \leq i \leq 3, 1 \leq j \leq n\} \). We prove \( K_{3,n} \) does not have a 3-VDIET coloring when \( n = 3, 4 \).

If not, let \( g \) be a 3-VDIET coloring of \( K_{3,n} \) using colors 1, 2, 3. Then \( |C(u_i)| \geq 2 \), \( i = 1, 2, 3 \). (Otherwise we assume \( C(u_1) = \{1\} \). Then \( 1 \in C(v_j), j = 1, 2, \ldots, n \). Thus \( C(v_1), C(v_2), \ldots, C(v_n) \) are not available for any vertex and at most one of them is an empty set. Therefore there are at most \( 2^3 - 1 - 2 = 5 \) nonempty subsets of \( \{1, 2, 3\} \) which can be the color sets of vertices \( u_1, u_2, u_3, v_1, v_2, \ldots, v_n \). Five subsets cannot distinguish \( n + 3 \) vertices when \( n = 3, 4 \), this is a contradiction.)

Furthermore, \( |C(v_j)| \geq 2, j = 1, 2, \ldots, n \). (Otherwise we assume \( C(v_1) = \{1\} \). Then \( 1 \in C(u_i), i = 1, 2, 3 \). Thus \( C(u_1), C(u_2), C(u_3) \) are not available for any vertex and at most one of them is an empty set. Therefore there are at most \( 2^3 - 1 - 2 = 5 \) nonempty subsets of \( \{1, 2, 3\} \) which can be the color sets of vertices \( u_1, u_2, u_3, v_1, v_2, \ldots, v_n \). Five subsets cannot distinguish \( n + 3 \) vertices when \( n = 3, 4 \), this is a contradiction.)

So three 1-subsets of \( \{1, 2, 3\} \) are not available for any vertex, the remaining 4 nonempty subsets of \( \{1, 2, 3\} \) cannot distinguish \( n + 3 \) vertices when \( n = 3, 4 \), this is a contradiction. Therefore, \( \chi_{\text{et}}(K_{3,n}) \geq 4 \) when \( n = 3, 4 \).

In the following we give a 4-VDIET coloring of \( K_{3,n} \) using colors 1, 2, 3, 4 when \( 3 \leq n \leq 9 \).

Let \( u_1, u_2, u_3 \) receive color 4. Suppose \( S_1 = (\{3\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}) \) and let \( D(v_j) \) be the \( i \)-th term of \( S_1, i = 1, 2, \ldots, n \). Let \( v_1 \) and its incident edges receive color 3, let \( v_2, u_3v_2 \) receive color 1 and \( u_1v_2, u_2v_2 \) receive color 2.

For \( D(v_j) = \{a, b\} \), \( 3 \leq j \leq n \), \( a < b \), we assign \( a \) to \( u_1v_j \) and \( v_j \), assign \( b \) to \( u_2v_j \) and \( u_3v_j \).

For \( D(v_j) = \{a, b, c\} \), \( a < b < c \), we assign \( a, b, c \) to \( u_1v_j, u_2v_j, u_3v_j \) respectively and assign \( b \) to \( v_j \).
Then \( C(u_1) = \{1, 2, 3, 4\}, \ C(u_2) = \{2, 3, 4\}, \ C(u_3) = \{1, 3, 4\} \) and \( C(v_j) = D(v_j), j = 1, 2, \ldots, n \) with respect to the above coloring. Thus the above coloring is a VDIET coloring of \( K_{3,n}(3 \leq n \leq 9) \) using 4 colors. 

**Theorem 16.** \( \chi^{ie}_{vt}(K_{4,n}) = \)

\[
\begin{align*}
4, & \text{ when } 4 \leq n \leq 7; \\
5, & \text{ when } 8 \leq n \leq 23; \\
6, & \text{ when } 24 \leq n \leq 55; \\
7, & \text{ when } 56 \leq n \leq 58; \\
k, & \text{ when } (k-1) + \cdots + (k-1) - 4 < n \\
& \leq (k) + \cdots + (k) - 4, \ k \geq 7.
\end{align*}
\]

**Proof.** It is easy to verify the theorem is valid in each case when \( n \geq 8 \) by Theorem 5, 6, 8 respectively. Now we consider the case \( 4 \leq n \leq 7 \).

It is obvious \( \chi^{ie}_{vt}(K_{4,n}) \geq \xi(K_{4,n}) = 4, \) when \( 4 \leq n \leq 7 \).

In the following we give a 4-VDIET coloring of \( K_{4,n} \) using colors \( 1, 2, 3, 4 \) when \( 4 \leq n \leq 7 \). Let \( V(K_{4,n}) = \{u_1, u_2, u_3, u_4, v_1, v_2, \ldots, v_n\} \) and \( E(K_{4,n}) = \{u_iv_j : i = 1, 2, 3, 4; j = 1, 2, \ldots, n\} \).

Let \( u_1, u_2, u_3, u_4 \) receive color 4. Suppose \( S_2 = \{\{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \) and let \( D(v_i) \) be the \( i \)-th term of \( S_2, i = 1, 2, \ldots, n \). Let \( v_i \) receive the minimum number of \( D(v_i), i = 1, 2, \ldots, n \).

For \( D(v_j) = \{j, 4\}, j = 1, 2, 3 \), we assign color 4 to \( u_iv_j \) and color 4 to \( u_iv_j, i = 1, 2, 3, 4, i \neq j \).

For \( D(v_j) = \{a, b\}, 4 \leq j \leq n, a < b \), we assign color \( b \) to all edges \( u_iv_j \) if \( i \neq b \) and color \( a \) to its remaining incident edge \( u_iv_j \).

For \( D(v_j) = \{1, 2, 3\}, \) we assign color 2 to \( u_iv_j \) if \( i \neq 2 \) and assign color 3 to \( u_iv_j \).

Then \( C(u_i) = \{1, 2, 3, 4\} \setminus \{i\}, i = 1, 2, 3, C(u_4) = \{1, 2, 3, 4\} \) and \( C(v_j) = D(v_j), j = 1, 2, \ldots, n \) with respect to the above coloring. Thus the above coloring is a 4-VDIET coloring of \( K_{4,n}, 4 \leq n \leq 7 \). 

**Theorem 17.** \( \chi^{ie}_{vt}(K_{5,n}) = \)

\[
\begin{align*}
5, & \text{ when } 6 \leq n \leq 21; \\
6, & \text{ when } 22 \leq n \leq 53; \\
7, & \text{ when } 54 \leq n \leq 117; \\
8, & \text{ when } 118 \leq n \leq 121; \\
k, & \text{ when } (k-1) + \cdots + (k-1) - 5 < n \\
& \leq (k) + \cdots + (k) - 5, \ k \geq 8.
\end{align*}
\]

**Proof.** By Theorem 5, 6, 8 respectively we know the theorem is valid in each case.
Theorem 18. $\chi^{ie}_{vt}(K_{6,n}) = \left\{ \begin{array} {ll}
5, & \text{when } 6 \leq n \leq 19; \\
6, & \text{when } 20 \leq n \leq 51; \\
7, & \text{when } 52 \leq n \leq 115; \\
8, & \text{when } 116 \leq n \leq 243; \\
9, & \text{when } 244 \leq n \leq 248; \\
k, & \text{when } \left(\binom{k}{1} + \cdots + \binom{k-1}{7}\right) - 6 < n \\
\leq \left(\binom{k}{1} + \cdots + \binom{k}{7}\right) - 6, & k \geq 9.
\end{array} \right.$

Proof. By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when $n \geq 20$. Now we consider the case $6 \leq n \leq 19$.

Let $V(K_{6,n}) = \{u_1, u_2, \ldots, u_6, v_1, v_2, \ldots, v_n\}$ and $E(K_{6,n}) = \{u_i v_j : 1 \leq i \leq 6, 1 \leq j \leq n\}$. We prove $K_{6,n}$ does not have a 4-VDIET coloring when $6 \leq n \leq 9$. If not, suppose $g$ is a 4-VDIET coloring of $K_{6,n} (6 \leq n \leq 9)$ using colors 1, 2, 3, 4. Then $|C(u_i)| \geq 2, i = 1, 2, \ldots, 6$. (Otherwise we assume $C(u_1) = \{1\}$. Then $1 \in C(v_j), j = 1, 2, \ldots, n$. Thus $\overline{C}(u_1), \overline{C}(u_2), \ldots, \overline{C}(u_6)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^4 - 1 - 5 = 10$ nonempty subsets of $\{1, 2, 3, 4\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_6, v_1, v_2, \ldots, v_n$. These subsets cannot distinguish $n + 6$ vertices when $6 \leq n \leq 9$, this is a contradiction.)

Furthermore, $|C(v_j)| \geq 2, j = 1, 2, \ldots, n$. (Otherwise we assume $C(v_1) = \{1\}$, then $1 \in C(u_i), i = 1, 2, \ldots, 6$. Thus $\overline{C}(u_1), \overline{C}(u_2), \ldots, \overline{C}(u_6)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^4 - 1 - 5 = 10$ nonempty subsets of $\{1, 2, 3, 4\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_6, v_1, v_2, \ldots, v_n$. These subsets cannot distinguish $n + 6$ vertices when $6 \leq n \leq 9$, this is a contradiction.) So four 1-subsets of $\{1, 2, 3, 4\}$ are not available for any vertex, the remaining 11 nonempty subsets of $\{1, 2, 3, 4\}$ cannot distinguish $n + 6$ vertices when $6 \leq n \leq 9$, this is a contradiction. Therefore, $\chi^{vt}_{vt}(K_{6,n}) \geq 5$ when $6 \leq n \leq 9$.

In the following we give a 5-VDIET coloring of $K_{6,n}$ using colors 1, 2, 3, 4, 5 when $6 \leq n \leq 19$.

Let $u_1, u_2, \ldots, u_6$ receive color 5. Suppose $S_3 = \{\{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4\}\}$ and let $D(v_i)$ be the $i$-th term of $S_3, i = 1, 2, \ldots, n$. Let $D(u_i) = \{1, 2, 3, 4, 5\} \setminus \{i\}, i = 1, 2, 3, 4$, $D(u_5) = \{1, 2, 3, 4, 5\}$ and $D(u_6) = \{1, 2, 5\}$.

Let $u_i v_j (i = 1, 2, 3, 4), u_6 v_3$ and $u_6 v_4$ receive color 5. Let $v_j$ and the other incident edges of $v_j$ receive color $j, j = 1, 2, 3, 4$.

For $D(v_1) = \{a, b\}, 5 \leq j \leq n, a < b$, we assign $b$ to $u_i v_j$ if $b \in D(u_i)$, assign $a$ to $v_j$ and its remaining incident edges.
For $D(v_j) = \{a, b, c\}, \{b, c\} \neq \{3, 4\}$, $a < b < c$, we assign $b$ to $u_iv_j$ if $b \in D(u_i)$, assign $c$ to $u_iv_j$ if $b \notin D(u_i)$, and assign $a$ to $v_j$.

For $D(v_j) = \{a, 3, 4\}$, $a = 1, 2$, we assign $a$ to $u_iv_j$ if $a \in D(u_i)$, assign $3$ to $u_iv_j$ if $a \notin D(u_i)$, and assign $4$ to $v_j$.

For $D(v_j) = \{1, 2, 3, 4\}$, we assign $3$ to $u_iv_j$ if $3 \in D(u_i)$, assign $4$ to $u_3v_j$, assign $2$ to $u_6v_j$, and assign $1$ to $v_j$.

Then $C(u_i) = D(u_i), 1 \leq i \leq 6$ and $C(v_j) = D(v_j), 1 \leq j \leq n$ with respect to the above coloring. Thus the above coloring is a 5-VDIET coloring of $K_{6,n}, 6 \leq n \leq 19$.

Theorem 19. $\chi_{vit}^e(K_{7,n}) = \begin{cases} 
5, & \text{when } 7 \leq n \leq 17; \\
6, & \text{when } 18 \leq n \leq 49; \\
7, & \text{when } 50 \leq n \leq 113; \\
8, & \text{when } 114 \leq n \leq 241; \\
9, & \text{when } 242 \leq n \leq 497; \\
10, & \text{when } 498 \leq n \leq 503; \\
k, & \text{when } \binom{k-1}{1} + \cdots + \binom{k-1}{7} - 7 < n \leq \binom{k}{1} + \cdots + \binom{k}{8} - 7, k \geq 10.
\end{cases}$

Proof. By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when $n \geq 50$. Now we consider the case $n \leq 49$.

$$\xi(K_{7,n}) = \begin{cases} 
4, & \text{when } n = 7, 8; \\
5, & \text{when } 9 \leq n \leq 24; \\
6, & \text{when } 25 \leq n \leq 49.
\end{cases}$$

Let $V(K_{7,n}) = \{u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n\}$ and $E(K_{7,n}) = \{u_iv_j : 1 \leq i \leq 7, 1 \leq j \leq n\}$.

We prove $K_{7,n}$ does not have a 4-VDIET coloring when $n = 7, 8$. If not, suppose $g$ is a 4-VDIET coloring of $K_{7,n}, (n = 7, 8)$ using colors $1, 2, 3, 4$. Then $|C(u_i)| \geq 2, i = 1, 2, \ldots, 7$. Otherwise we assume $C(u_1) = \{1\}$. Then $1 \in C(v_j), j = 1, 2, \ldots, n, n = 7, 8$. Thus $C(v_1), C(v_2), \ldots, C(v_n)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^4 - 1 - 6 = 9$ nonempty subsets of $\{1, 2, 3, 4\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. These subsets cannot distinguish 14 or 15 vertices, this is a contradiction.

Furthermore, $|C(v_j)| \geq 2, j = 1, 2, \ldots, n, n = 7, 8$. Otherwise we assume $C(v_1) = \{1\}$. Then $1 \in C(u_i), i = 1, 2, \ldots, 7$. Thus $C(u_1), C(u_2), \ldots, C(u_7)$ are not available for any vertex and at most one of them is an empty set. Therefore there are at most $2^4 - 1 - 6 = 9$ nonempty subsets of $\{1, 2, 3, 4\}$ which can be the color sets of vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. These subsets cannot distinguish 14 or 15 vertices, this is also a contradiction.) So four 1-subsets of $\{1, 2, 3, 4\}$ are not available for any vertex, the remaining 11 nonempty subsets of
{1, 2, 3, 4} cannot distinguish 14 or 15 vertices, this is a contradiction. Therefore, \( \chi_{5}^{c}(K_{7,n}) \geq 5 \) when \( n = 7, 8 \).

In the following we give a 5-VDIET coloring of \( K_{7,n} \) using colors 1, 2, 3, 4, 5 when \( 7 \leq n \leq 17 \).

Let \( u_{1}, u_{2}, \ldots, u_{7} \) receive color 5. Suppose \( S_{1} = \{ \{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}, \{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,2,5\}, \{1,3,4\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{3,4,5\}, \{1,2,3,4\} \} \) and let \( D(u_{i}) \) be the \( i \)-th term of \( S_{1}, i = 1, 2, \ldots, n \).

Let \( D(u_{i}) = \{1, 2, 3, 4, 5\} \setminus \{i\}, i = 1, 2, 3, 4, D(u_{5}) = \{1, 3, 5\}, D(u_{6}) = \{2, 4, 5\} \) and \( D(u_{7}) = \{1, 2, 3, 4, 5\} \).

Let \( u_{1}v_{1} \) and \( u_{6}v_{1} \) receive color 5, \( v_{1} \) and its other incident edges receive color 1. Let \( u_{2}v_{2} \) and \( u_{5}v_{2} \) receive color 5, \( v_{2} \) and its other incident edges receive color 2. Let \( u_{3}v_{3} \) and \( u_{6}v_{3} \) receive color 5, \( v_{3} \) and its other incident edges receive color 3.

Let \( u_{4}v_{4} \) and \( u_{5}v_{4} \) receive color 5, \( v_{4} \) and its other incident edges receive color 4.

For \( D(v_{j}) = \{a, b\}, 5 \leq j \leq n, a \neq b \), we assign \( b \) to \( u_{i}v_{j} \) if \( b \in D(u_{i}) \), assign \( a \) to \( v_{j} \) and its remaining incident edges.

For \( D(v_{j}) = \{a, b, c\}, \{a, b, c\} \neq \{1, 2, 4\}, a \neq b \neq c \), we assign \( b \) to \( u_{i}v_{j} \) if \( b \notin D(u_{i}) \), assign \( a \) to \( v_{j} \).

For \( D(v_{j}) = \{1, 2, 4\} \), we assign 1 to \( u_{i}v_{j} \) if \( 1 \in D(u_{i}) \), assign 2 to \( u_{i}v_{j} \) if \( 1 \notin D(u_{i}) \), and assign 4 to \( v_{j} \).

For \( D(v_{j}) = \{1, 2, 3, 4\} \), we assign 2 to \( u_{i}v_{j} \) if \( 2 \in D(u_{i}) \), assign 4, 3, 1 to \( u_{2}v_{j}, u_{3}v_{j}, v_{j} \) respectively.

Then \( C(u_{i}) = D(u_{i}), 1 \leq i \leq 7 \) and \( C(v_{i}) = D(v_{i}), j = 1, 2, \ldots, n \) with respect to the above coloring. Thus the above coloring is a 5-VDIET coloring of \( K_{7,n}, 7 \leq n \leq 17 \).

We prove \( K_{7,n} \) does not have a 5-VDIET coloring when \( 18 \leq n \leq 24 \). If not, suppose \( g \) is a 5-VDIET coloring of \( K_{7,n}(18 \leq n \leq 24) \) using colors 1, 2, 3, 4, 5. First we give four claims as follows.

**Claim 20.** \(|C(u_{i})| \geq 2, i = 1, 2, \ldots, 7\).

**Proof.** Suppose the claim is not true, without loss of generality we assume \( C(u_{1}) = \{1\} \). Then \( 1 \in C(v_{j}), j = 1, 2, \ldots, n, 18 \leq n \leq 24 \). Thus \( \overline{C}(v_{1}), \overline{C}(v_{2}), \ldots, \overline{C}(v_{n}) \) are not available for any vertex and at most one of them is an empty set. Therefore there are at most \( 2^{5} - 1 - 17 = 14 \) nonempty subsets of \( \{1, 2, 3, 4, 5\} \) which can be the color sets of vertices \( u_{1}, u_{2}, \ldots, u_{7}, v_{1}, \ldots, v_{n} \). These subsets cannot distinguish \( n + 7 \) vertices when \( 18 \leq n \leq 24 \), this is a contradiction. \[ \square \]

**Claim 21.** \(|C(v_{i})| \geq 2, j = 1, 2, \ldots, n, 18 \leq n \leq 24\).

**Proof.** Suppose the claim is not true, without loss of generality we assume \( C(v_{1}) = \{1\} \). Then \( 1 \in C(u_{i}), i = 1, 2, \ldots, 7 \). Thus \( \overline{C}(u_{1}), \overline{C}(u_{2}), \ldots, \overline{C}(u_{7}), \{g(u_{1}), g(u_{2}), \ldots, g(u_{7})\} \) are not available for any vertex and at most one of them
is an empty set. Therefore there are at most $2^5 - 1 - 7 = 24$ nonempty subsets of \{1, 2, 3, 4, 5\} which can be the color sets of vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. These subsets cannot distinguish $n + 7$ vertices when $18 \leq n \leq 24$, this is also a contradiction.

\[\square\]

**Claim 22.** $C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) = \emptyset$.

**Claim 23.** $C(v_1) \cap C(v_2) \cap \cdots \cap C(v_n) = \emptyset$, $18 \leq n \leq 24$.

The proofs of Claim 22 and Claim 23 are analogous to the proofs of Claim 11 and Claim 12 in Theorem 8, respectively.

By Claims 20 and 21, five 1-subsets of \{1, 2, 3, 4, 5\} are not available for any vertex. The remaining 26 nonempty subsets of \{1, 2, 3, 4, 5\} cannot distinguish $n + 7$ vertices when $20 \leq n \leq 24$, this is a contradiction. So we assume $n = 18, 19$ in the following.

Let $t = |\{g(u_1), g(u_2), \ldots, g(u_7)\}|$, and $\{g(u_1), g(u_2), \ldots, g(u_7)\} = \{1, 2, \ldots, t\}$, by Claim 22 and Claim 23, we know that $t = 2$ or $t = 3$.

**Case 1.** $t = 2$, \{f(u_1), f(u_2), \ldots, f(u_7)\} = \{1, 2\}$. Of course \{1, 2\} $\notin$ \{C(v_1), C(v_2), \ldots, C(v_n)\}. If \{1, 2\} $\in$ \{C(v_1), C(v_2), \ldots, C(v_7)\}, then $1 \in C(v_j)$ or $2 \in C(v_j), j = 1, 2, \ldots, n$. Thus \{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\} cannot be the color sets of any vertices. Moreover, five 1-subsets are not available for any vertex. Then at most $2^5 - 1 - 5 - 4 = 22$ nonempty subsets of \{1, 2, 3, 4, 5\} are available for the vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. This is a contradiction because 22 subsets cannot distinguish 25 (when $n = 18$) or 26 (when $n = 19$) vertices. So \{1, 2\} is not available for any vertex.

If $|C(u_i)| \geq 3, i = 1, 2, \ldots, 7$, then $C \subseteq C(u_1), C(u_2), \ldots, C(u_7)$ cannot be the color sets of any vertices because there are 5 colors in all. At most one of $C \subseteq C(u_1), C(u_2), \ldots, C(u_7)$ is an empty set, so there are at most $2^5 - 1 - 6 - 1 = 24$ nonempty subsets of \{1, 2, 3, 4, 5\} are available for the vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. This is a contradiction because 24 subsets cannot distinguish 25 (when $n = 18$) or 26 (when $n = 19$) vertices.

Therefore, there exists a vertex $u_{i_0}$ with $|C(u_{i_0})| = 2$. Since \{1, 2\} is not available for any vertex, so without loss of generality, we assume $C(u_{i_0}) = \{1, 3\}$, then $1 \in C(v_j)$ or $3 \in C(v_j), j = 1, 2, \ldots, n$. Thus \{4, 5\} is not available for any vertex. Furthermore, \{1, 2\} and five 1-subsets are not available for any vertex. There are at most $2^5 - 1 - 5 - 2 = 24$ nonempty subsets of \{1, 2, 3, 4, 5\} are available for the vertices $u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n$. This is a contradiction because 24 subsets cannot distinguish 25 (when $n = 18$) or 26 (when $n = 19$) vertices.

So $K_{7,n}(n = 18, 19)$ does not have a 5-VDIE coloring in this case.

**Case 2.** $t = 3$, \{f(u_1), f(u_2), \ldots, f(u_7)\} = \{1, 2, 3\}$. By Claim 23, $|\{f(v_1), f(v_2), \ldots, f(v_n)\}| \geq 2$, so \{f(v_1), f(v_2), \ldots, f(v_n)\} = \{4, 5\}. Then \{4, 5\} is not
the color set of any vertex \( u_i, i = 1, 2, \ldots, 7 \). If \( \{4, 5\} \in \{C(u_i), C(v_2), \ldots, C(v_n)\} \), then \( 4 \in C(u_i) \) or \( 5 \in C(u_i), i = 1, 2, \ldots, 7 \). Thus \( \{1, 2, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \) cannot be the color sets of any vertex. Moreover, five 1-subsets are not available for any vertex. Then at most \( 2^5 - 1 - 5 - 4 = 22 \) nonempty subsets of \( \{1, 2, 3, 4, 5\} \) are available for the vertices \( u_1, u_2, \ldots, v_7, v_1, v_2, \ldots, v_n \). This is a contradiction because 22 subsets cannot distinguish 25 (when \( n = 18 \)) or 26 (when \( n = 19 \)) vertices. So \( \{4, 5\} \) is not available for any vertex.

If \( |C(v_j)| \geq 3, j = 1, 2, \ldots, n \), then \( C(v_i), C(v_2), \ldots, C(v_n) \) cannot be the color sets of any vertex because there are 5 colors in all. At most one of them is available for the vertices \( u_1, u_2, \ldots, v_7, v_1, v_2, \ldots, v_n \). This is a contradiction because these subsets cannot distinguish 25 (when \( n = 18 \)) or 26 (when \( n = 19 \)) vertices.

Therefore, there exists a vertex \( v_{j_0} \) with \( |C(v_{j_0})| = 2 \). Since \( \{4, 5\} \) is not available for any vertex, so without loss of generality, we assume \( C(v_{j_0}) = \{1, 4\} \). Then \( 1 \in C(u_i) \) or \( 4 \in C(u_i), i = 1, 2, \ldots, 7 \). Thus \( \{2, 3\} \) is not available for any vertex. Moreover, \( \{4, 5\} \) and five 1-subsets are not available for any vertex. There are at most \( 2^5 - 1 - 5 - 2 = 24 \) nonempty subsets are available for the vertices \( u_1, u_2, \ldots, u_7, v_1, v_2, \ldots, v_n \). This is a contradiction because 24 subsets cannot distinguish 25 (when \( n = 18 \)) or 26 (when \( n = 19 \)) vertices.

So \( K_{7,n} (n = 18, 19) \) does not have a 5-VDIET coloring.

Therefore, \( \chi^{opt}_{vd}(K_{7,n}) \geq 6 \) when \( 18 \leq n \leq 49 \).

In the following we give a 6-VDIET coloring of \( K_{7,n} \) using colors 1, 2, 3, 4, 5, 6 when \( 18 \leq n \leq 49 \).

Arrange all 49 subsets of \( \{1, 2, 3, 4, 5, 6\} \) except for \( \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\} \), \( \{4, 5\} \), \( \{2, 3, 4, 5, 6\} \), \( \{1, 3, 4, 5, 6\} \), \( \{1, 2, 4, 5, 6\} \), \( \{1, 2, 3, 5, 6\} \), \( \{1, 2, 3, 4, 6\} \), \( \{1, 2, 3, 4, 5, 6\} \), \( \{1, 2, 3, 4, 5, 6\} \), \( \{1, 2, 3, 4, 5, 6\} \), \( \{1, 2, 3, 4, 5, 6\} \), \( \{1, 2, 3, 4, 5, 6\} \) into a sequence \( S_5 \) such that the first 5 terms are \( \{1, 6\}, \{2, 6\}, \{3, 6\}, \{4, 6\}, \{5, 6\} \) respectively. Let \( D(v_i) \) be the \( j \)-th term of \( S_5, j = 1, 2, \ldots, n \). Let \( D(u_i) = \{1, 2, 3, 4, 5, 6\} \setminus \{i\}, i = 1, 2, 3, 4, 5 \), \( D(u_i) = \{1, 2, 3, 4, 5, 6\} \setminus \{u_7\} = \{1, 2, 3, 4, 6\} \).

Let \( u_1, u_2, \ldots, u_7 \) receive color 6. Let \( v_j \) receive color \( j, j = 1, 2, \ldots, 5 \). Let \( u_i v_i \) receive color 6, \( i = 1, 2, \ldots, 5 \). Let \( u_i v_j \) receive color \( j, i = 1, 2, \ldots, 6, j = 1, 2, \ldots, 5, i \neq j \). Let \( u_1 v_1, u_7 v_2, u_7 v_3, u_7 v_4 \) and \( u_7 v_5 \) receive colors 1, 2, 3, 6 and 6 respectively.

For \( D(v_i) = \{a, b\}, 6 \leq j \leq n, a < b, \) we assign \( b \) to \( u_i v_j \) if \( b \in D(u_i) \), assign \( a \) to \( v_j \) and its remaining incident edges.

For \( D(v_j) = \{a, 4, 5\}, 1 \leq a \leq 3 \), we assign \( 5 \) to \( v_j \), \( a \) to \( u_i v_j \) if \( a \in D(u_i) \), assign \( 4 \) to \( u_i v_j \) otherwise.

For \( D(v_j) = \{a, b, c\}, a < b < c, \{b, c\} \neq \{4, 5\} \), we assign \( a \) to \( v_j \), \( b \) to \( u_i v_j \) if \( b \in D(u_i) \), assign \( c \) to \( u_i v_j \) otherwise.

For \( D(v_j) = \{a, b, c, d\}, a < b < c < d \), we assign \( a \) to \( v_j \), \( b \) to \( u_i v_j \) if
b ∈ D(u_i), i ≠ 6, assign c to u_iv_j if b ∉ D(u_i), c ∈ D(u_i), i ≠ 6, and assign d to the remaining incident edges of v_j.

For D(v_j) = {1, 2, 3, 4, 5}, we assign 1 to v_j, assign 2, 3, 4, 5 to u_3v_j, u_4v_j, u_5v_j, u_6v_j respectively and assign 3 to the remaining incident edges of v_j.

Then C(u_i) = D(u_i), 1 ≤ i ≤ 7 and C(v_j) = D(v_j), j = 1, 2, . . . , n with respect to the above coloring. Thus the above coloring is a 6-VDIET coloring of K_7,n, 24 ≤ n ≤ 49.

**Theorem 24.** Let K_n be the complete graph of order n (n ≥ 3). Then χ_{vt}(K_n) = n.

**Proof.** As any two vertices in K_n must receive different colors under an arbitrary VDIET coloring, therefore χ_{vt}(K_n) ≥ n. Of course we may be able to show that χ_{vt}(K_n) = n by giving a VDIET coloring of K_n using n colors 1, 2, . . . , n as follows. Assign colors 1, 2, . . . , n to vertices v_1, v_2, . . . , v_n of K_n respectively and then let all edges receive the same color 1. ■

From the results obtained in this paper, we know that for any graph G discussed in this paper except K_n (n ≥ 6), we have χ_{vt}(G) = ξ(G) or ξ(G) + 1. So we propose the following conjectures.

**Conjecture 25.** For a simple graph G, if its (proper vertex coloring) chromatic number χ(G) ≤ 4, then we have χ_{vt}(G) = ξ(G) or ξ(G) + 1.

**Conjecture 26.** For a simple graph G, we have χ_{vt}(G) ≤ max{ξ(G) + 1, χ(G)}.

**Conjecture 27.** Let s be the minimum positive integer such that 2^s − 1 ≥ 3m. When 2^r − 2m − 1 < n ≤ 2^{r+1} − 2m − 1, we have χ_{vt}(K_{m,n}) = r + 1, where r = s, s + 1, . . . , m − 2, s ≤ m − 2.

**Acknowledgement**

The authors would like to thank the referees for their valuable comments and helpful suggestions.

**References**


Received 11 October 2010
Revised 11 July 2011
Accepted 5 March 2012