Abstract

Significant values of a combinatorial count need not fit the recurrence for the count. Consequently, initial values of the count can much outnumber those for the recurrence. So is the case of the count, $G_l(n)$, of distance-$l$ independent sets on the cycle $C_n$, studied by Comtet for $l \geq 0$ and $n \geq 1$ [sic]. We prove that values of $G_l(n)$ are $n$th power sums of the characteristic roots of the corresponding recurrence unless $2 \leq n \leq l$. Lucas numbers $L(n)$ are thus generalized since $L(n)$ is the count in question if $l = 1$. Asymptotics of the count for $1 \leq l \leq 4$ involves the golden ratio (if $l = 1$) and three of the four smallest Pisot numbers inclusive of the smallest of them, plastic number, if $l = 4$. It is shown that the transition from a recurrence to an OGF, or back, is best presented in terms of mutually reciprocal (shortly: co-reciprocal) polynomials. Also the power sums of roots (i.e., moments) of a polynomial have the OGF expressed in terms of the co-reciprocal polynomial.

Keywords: distance independent set, Lucas numbers, Pisot numbers, power sums, generating functions, (co-) reciprocal polynomials.

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1. Introduction

In what follows we restrict our study to connected $n$-vertex graphs, the path $P_n$ with $n \geq 1$ and the cycle $C_n$ with $n \geq 2$, which are simple, with exception that $C_2$ will stand for the 2-vertex multigraph $2K_2$, the 2-cycle. Additionally, $C_1$ will stand for the (1-vertex) loop-graph. The letter $l$ stands for a nonnegative integer.
Our aim is to study the numbers, say $F_l(n)$ and $G_l(n)$, of $l$-independent sets (inclusive of the empty set) on the path $P_n$ and the cycle $C_n$, respectively.

The distance between any two vertices $x$ and $y$ in a graph $G$ is the length of a shortest $x$–$y$ path of $G$. A set $S$ (possibly empty) is called $l$-independent in $G$ if $S$ comprises vertices of $G$ and any two elements of $S$ are distance at least $l + 1$ apart. In other words, if an $l$-independent set $S$ includes distinct vertices $x$ and $y$ then every $x$–$y$ path of $G$ includes $l$ or more vertices which do not belong to $S$. Consequently, each vertex subset of $G$ is 0-independent. Moreover, 1-independent coincides with independent.

The numbers $F_l(n)$ and $G_l(n)$, denoted respectively by $F(n+l,l)$ and $G(n,l)$, appear in Comtet [4, p. 46]. Their OGFs (ordinary generating functions) are presented, too, though the case of $G_l$ for any $l > 1$ is questionable, see Remark 4 in Section 5 below. Moreover, closed formulas for the corresponding numbers, $f_l(n,p)$ and $g_l(n,p)$, of $l$-independent sets of cardinality $p$ are presented in Comtet [4, pp. 21,24], namely

$$f_l(n,p) = \binom{n-(p-1)l}{p} \text{ and } g_l(n,p) = \binom{n-pl}{p}.$$  

The formula for $f_l$ is credited to Gergonne (1812) and Muir (1902) and that for $g_l$ to Kaplansky (1943), but the parameter $l$ therein is due to Comtet since independent sets only, i.e. for $l = 1$, (on $F_n$ and $C_n$) are counted by Kaplansky.

All the four sequences of numbers and the two formulas in question, though for $l = 1$ only, appeared earlier in Berge’s book, see [2, pp. 31–32]. Clearly,

$$F_l(n) = \sum_{p \geq 0} f_l(n,p) \text{ and } G_l(n) = \sum_{p \geq 0} g_l(n,p).$$

Note that for $l = 0$ the four numbers are pairwise $2^n$ (= $F_0(n) = G_0(n)$) and $\binom{n}{p}$ (= $f_0(n,p) = g_0(n,p)$). It is known that for $l = 1$ the number $F_1(n)$ is the shifted Fibonacci number $F_{n+2}$, as in Sloane [15], under the assumption that Fibonacci numbers $F_n$ begin at 0, 1 (with $F_0 = 0$). On the other hand, $G_1(n) = L(n)$, which is the $n$th Lucas number (as noted in [10], but not in Comtet [4], and called a corrected Fibonacci number in Berge [2]), with two initial values 2 (= $L(0)$), 1.

All the four (sequences of) numbers (but with distance bound $l$ expressed in terms of $k = l + 1$) are presented in [7]. Also the linear recurrence for $F(k,n)$ (= $F_l(n)$ in our notation) appears in [7].

Our main objective is the study of the numbers $G_l(n)$ via the corresponding recurrence and its characteristic roots. The known recurrence for $F_l(n)$ is recalled (with a simplified proof) because it considerably simplifies our reasoning. We show that both $F_l(n)$ and $G_l(n)$, on denoting them by $a(n)$, satisfy the same 3-term linear homogeneous recurrence

$$u(n) = u(n-1) + u(n-l-1).$$

In fact, $G_l(n)$ satisfies the recurrence (but only for $n \geq 2l + 2$ if $l \geq 2$) and generalizes (includes) integer sequences: powers of 2 ($l = 0$) and Lucas numbers.
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$L(n) (l = 1)$, where $L(n)$ is the sum of $n$th powers of the two characteristic roots (including the golden ratio) of the recurrence (1) with $l = 1$. Our main result is a simple proof that in the remaining case of $l \geq 2$, $G_l(n)$ is the sum of $n$th powers of all $l + 1$ characteristic roots unless $2 \leq n \leq l$. Hence we derive both the asymptotic equivalent of $G_l(n)$ for any $l$ and, for small $l$ only, a simple formula in terms of nearest integer function $\floor{\cdot}$. Moreover, the related recent formula for the number of Hamilton cycles in the square of a cycle is discussed. Rational OGF for the sequence of moments (defined to be power sums of roots) of any polynomial is announced.

2. Distance-independent Sets

We shall use classical setting for the problem in question. Namely, as in Comtet, the path $P_n$ is represented by the integer interval $[n] := \{1, 2, \ldots, n\}$ for $n \geq 1$ and the cycle $C_n$ by the cyclic group $\mathbb{Z}_n =: [\tilde{n}]$, with elements $0, 1, \ldots, n - 1$, for $n \geq 1$, too.

**Theorem 1.** For any nonnegative integer $l \geq 0$, $F_l(n)$ and $G_l(n)$ stand for the counts of $l$-independent vertex subsets on the path $P_n$ and the cycle $C_n$, respectively. Then

(2) $F_l(n) = F_l(n - 1) + F_l(n - l - 1)$ for $n \geq l + 1$, with initial conditions

(3) $F_l(n) = n + 1$ for $n = 1, \ldots, l$, extended to $n = 0$ by $F_l(0) := 1$;

(4) $G_l(n) = G_l(n - 1) + G_l(n - l - 1)$ for $n \geq 2l + 2$ if $l \geq 2$,

and $n \geq l + 1$ if $l = 0, 1$, with initial conditions

(5) $G_l(0) := l + 1$ for $l \geq 0$, $G_l(1) := 1$ for $l \geq 1$,

(6) $G_l(n) = n + 1$ for $n = 2, 3, \ldots, 2l + 1$ if $l \geq 1$.

**Remark 2.** $G_1(1) := 1$ counts the empty subset only. This reflects the convention that the vertex (as well as the edge) of the loop graph is self-adjacent and therefore self-dependent.

**Proof.** Definitions concerning $n = 0, 1$ in (3) and (5) conveniently extend validity of the corresponding recurrence (2) and (4), though (4) for $l = 0, 1$ only. For $l = 0$, all equalities are clear, also in (2) and (4). Consequently, $F_0(n) = 2^n = G_0(n)$ for any admissible $n$. 

Therefore we assume that \( l \geq 1 \). Initial conditions (3) and (6) are easily seen.

Let us determine the number \( F_l(n) \) of \( l \)-independent subsets \( X \) of \([n]\) for \( n \geq l + 1 \geq 2 \). The subsets \( X \) containing \( n \) do not contain any of \( l \) integers \( n - 1, n - 2, \ldots, n - l \), and hence there are \( F_l(n - l - 1) \) of the sets \( X \); those not containing \( n \) amount to \( F_l(n - 1) \), whence (2) follows. Hence

\[
(7) \quad F_l(n) = F_l(n - 1) + F_l(n - l - 1) \quad \text{for } n \geq l + 1 \quad (\text{since } F_l(0) = 1).
\]

Assume that \( l = 1 \). Then the recurrence (4) holds for \( n = 2, 3 \) due to (5) since \( G_1(n) = n + 1 \) for \( n = 2, 3 \), see (6). It remains to determine the number \( G_l(n) \) of \( l \)-independent subsets of \([n]\) for any \( l \geq 1 \) and \( n > 2l + 1 \). Then the subsets which contain 0 do not contain any of \( 2l \) integers \( 1, 2, \ldots, l \) and \( n - 1, n - 2, \ldots, n - l \), whence there are \( F_l(n - 2l - 1) \) of the subsets. Similar statement is true if subsets contain any integer \( m \in [n] \). Therefore subsets, \( Y \), which contain any of \( l \) consecutive integers \( n - l + 1, n - l + 2, \ldots, n = 0 \), contain exactly one of them. Hence the class of sets \( Y \) splits into \( l \) parts of cardinality \( F(n - 2l - 1) \) each. On the other hand, remaining \( l \)-independent subsets contain none of those \( l \) integers. Hence there are \( F_l(n - l) \) of such subsets. Consequently,

\[
G_l(n) = F_l(n - l) + l \cdot F_l(n - 2l - 1) \quad \text{for } n \geq 2l + 2,
\]

where, by (7) with \( n \) replaced by \( n - l \),

\[
F_l(n - 2l - 1) = F_l(n - l) - F_l(n - l - 1) \quad \text{for } n \geq 2l + 1.
\]

On substituting,

\[
(8) \quad G_l(n) = (l + 1)F_l(n - l) - l \cdot F_l(n - l - 1),
\]

which holds not only for \( n \geq 2l + 2 \) but also for \( l + 1 \leq n \leq 2l + 1 \) due to the stated initial values of \( G_l \) and \( F_l \). Hence, first by (8) for \( n \geq 2l + 2 \),

\[
\begin{align*}
G_l(n - 1) + G_l(n - l - 1) & = (l + 1)(F_l(n - l - 1) + F_l(n - 2l - 1)) - l (F_l(n - l - 2) + F_l(n - 2l - 2)) \\
& = (l + 1)F_l(n - l) - l \cdot F_l(n - l - 1) \quad (\text{by } (7)), \\
& = G_l(n) \quad (\text{by } (8)),
\end{align*}
\]

which completes the proof.

\section{Cyclic Strong Independence}

Note that significant values of the count \( G_l(n) \), namely exactly those on short \( n \)-cycles with \( 2 \leq n \leq l \), do not fit the recurrence (4) (in case \( l \geq 2 \) only). We now modify those values so that the recurrence could hold for \( n \geq l + 1 \) with
We next show that the modified count comprises $n$th power sums of the $l+1$ characteristic roots of the recurrence for all $n \geq 0$ and $l \geq 0$. Let

$$G_l^*(n) = \begin{cases} 1 & \text{for } n = 2, \ldots, l \text{ with } l \geq 2, \\ G_l(n) & \text{otherwise}. \end{cases}$$

Proposition 3. The sequence $G_l^*(n)$ satisfies recurrence (1) for $n \geq l + 1$, with initial values as above.

Proof. In view of Theorem 1 it is enough to see the following. Assume that $l \geq 2$. Then for $l + 2 \leq n \leq 2l + 1$, due to (9) and (6), we have

$$G_l^*(n - 1) + G_l^*(n - l - 1) = G_l(n - 1) + 1 = n + 1 = G_l^*(n),$$

as required. For $n = l + 1$, we have $G_l^*(n) = (l + 1) + 1 = G_l^*(0) + G_l^*(n - 1)$, as required, too.

Hence and in regard to Remark 2 the following definition is motivated. A vertex subset $S$ of a (general) graph (or a cycle) $G$ is $l$-independent (or cyclically strong $l$-independent) in $G$ if $S$ is $l$-independent unless $l \geq 1$, the graph $G$ is a short cycle, $G = \mathbb{Z}_n$ with $1 \leq n \leq l$, and $|S| > 0$. Thus only the empty set is $l$-independent on a short cycle if $l \geq 1$. Therefore $G_l^*(n)$ is the count of such $l$-independent subsets on the $n$-cycle.

For other information on sequences $G_l^*(n)$, see sequence A000204 (Lucas numbers beginning with $L(1) = 1$) in [15] and comments therein on generalizations.

4. **Recurrence-OGF and Co-reciprocal Polynomials**

It is a good opportunity now to show how the notion of mutually reciprocal polynomials simplifies the procedure which leads from a given recurrence which is LinHomConst (linear homogeneous with constant coefficients) and complete (i.e., with initial values) to the corresponding OGF (and/or vice versa). Let

$$g(z) = \sum_{j=0}^{r} c_j z^j \in \mathbb{C}[z] \text{ with constant term } c_0 \neq 0$$

be a complex polynomial of positive degree $r$ and with nonzero roots only, possibly multiple. Then we say that the polynomial $f(z) := z^r g(z^{-1})$ is co-reciprocal for (or the reciprocal polynomial of) $g(z)$, and that polynomials $f(z)$ and $g(z)$ are co-reciprocal (or mutually reciprocal). These notions are not well-established in literature yet; e.g., ‘reciprocal’ in Andrews’ [1] means ‘self-reciprocal’. A self-reciprocal polynomial is invariant under reciprocation of the set of roots and so invariant is the set of roots itself. By the way, the minimal polynomial of the
golden ratio, \( h_1(x) := x^2 - x - 1 \) (see (13) with \( l = 1 \)), is not so invariant, but the reciprocation of its roots results in negating both of roots.

A polynomial \( f(x) \in \mathbb{C}[x] \) is said to be characteristic or in characteristic form if \( f(x) \) is monic, of positive degree, say \( r \), with nonzero roots, and with coefficient at \( x^{r-j} \) denoted by \( a_j \):

\[
(11) \quad f(x) = \sum_{j=0}^{r} a_j x^{r-j} \quad \text{with positive } r, a_r \neq 0 \text{ and } a_0 = 1.
\]

A polynomial \( Q(x) = \sum_{j=0}^{r} c_j z^j \) is said to be co-characteristic or in co-characteristic form if \( Q(x) \) is the reciprocal polynomial of a characteristic polynomial, that is, the co-reciprocal polynomial \( x^\deg Q(x) Q(1/x) \) is a characteristic polynomial. Then the constant term of \( Q(x) \), \( c_0 = 1 \). We say that a recurrence is a characteristic recurrence or is in the characteristic form if the recurrence is LinHomConst, with highest argument \( n \), the highest coefficient, say, \( c_0 = 1 \), and is as in (12) below.

Note that given a characteristic (order-\( r \)) recurrence (12), substitutions \( u(n-j) \leftarrow x^j \) in the left-hand side therein produce a polynomial, say \( Q(x) \), in co-characteristic form, and reciprocation of \( Q(x) \) gives a characteristic (degree-\( r \)) polynomial, \( f(x) \), which is characteristic polynomial of the recurrence, too. Therefore \( Q(x) \) is said to be the co-characteristic polynomial of the recurrence. On the other hand, \( f(x) \) is obtained straightforwardly by the substitutions \( u(n-j) \leftarrow x^{r-j} \) (instead of the former ones) provided that \( r \) is the order of the recurrence.

Going backwards from \( f(x) \) we arrive at the corresponding characteristic recurrence with \( f(x) \) as a characteristic polynomial of the recurrence. Passing on to the intermediate stage, the polynomial \( Q(x) \), simplifies hand calculations.

In this section it is assumed that a count/sequence \( u(n) \) is defined for \( n \geq n_1 \geq 0 \) where \( n_1 \) is an initial argument. Then \( u(j) := 0 \) for all integers \( j < n_1 \).

**PROCEDURE LinHomConstR-OGF**.

Input [A complete characteristic recurrence of order \( r \)]:

\[
(12) \quad \sum_{j=0}^{r} c_j u(n-j) = 0 \quad \text{for } n \geq k \text{ where a certain } k \geq r,
\]

with at least \( r \) initial values (of which last \( r \) ones are initial for the recurrence):

\[ u(n_1), u(n_1+1), \ldots, u(k-r), \ldots, u(k-1) \]

for some \( n_1 \leq k-r \), provided that \( c_j \) are constant coefficients, \( c_0 = 1 \) and \( c_r \neq 0 \).

Output [The OGF (possibly reducible), say]:

\[ \phi(x) = \frac{P(x)}{Q(x)} \], where \( Q(x) \) is the co-characteristic polynomial of the OGF,
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\[ Q(x) = \sum_{j=0}^{r} c_j x^j, \text{ with coefficients } c_j \text{ taken from the recurrence,} \]

\[ P(x) := Q(x) \cdot \phi(x) = Q(x) \sum_{j=n_1}^{k-1} u(j)x^j \mod x^k, \text{ a polynomial of degree less than } k. \]

Note that reducing the OGF (if possible) leads to an equivalent simpler recurrence, by using what follows.

The following converse procedure includes a recursive generation, see Stanley [16], of initial values of the count.

**PROCEDURE OGF-LinHomRec.**

Input [A rational function \( \Phi(x) := P(x)/Q(x) \) which is the irreducible OGF for \( u(n) \) where \( n \geq n_1 \geq 0 \). Let \( r = \deg Q(x) \), \( Q(x) = \sum_{j=0}^{r} c_j x^j \) with \( c_0 = Q(0) = 1 \), as above. Let \( b_j \) be coefficients of the numerator polynomial \( P(x) \), \( P(x) = \sum_{j=0}^{s} b_j x^j \) with \( \deg P(x) = s \).]

Output [The recurrence (LinHomConst and of the smallest possible order \( r \)) is obtainable from the co-characteristic polynomial \( Q(x) \):

\[ u(n) + \sum_{j=1}^{r} c_j u(n - j) = 0 \quad \text{for } n \geq \max(r + n_1, 1 + s). \]

The resulting recurrence is valid for \( n \geq \max(\deg Q(x) + n_1, 1 + \deg P(x)) \). Initial (and any) terms \( u(m) \) of the sequence \( u(n) \) can be found recursively on equating coefficients of \( x^m \) in the identity

\[ Q(x) \cdot \sum_{m \geq 0} u(m)x^m = P(x). \]

Consequently, values of \( u(n) \) (inclusive of the initial ones, for \( n_1 \leq n \leq \max(r + n_1 - 1, s) \), are found recursively for consecutive \( m = 0, 1, \ldots \) from

\[ u(m) + \sum_{j=1}^{\min(m,r)} c_j u(m - j) = b_m \]

where \( b_m = 0 \) for \( m < n_1 \) and for \( m > s = \deg P(x) \).]

5. OGF and Power Sums of Roots

The recurrences (2), (4), and (1) are LinHomConst (linear homogeneous, with constant coefficients) and of order \( l + 1 \) and are essentially the same. Their characteristic polynomial, say \( h_l(x) \), for \( x = z \in \mathbb{C} \), is

\[ h_l(z) = z^{l+1} - z^l - 1, \]

with all characteristic roots being nonzero.
We now find an OGF, say \( \Phi(x) = \Phi_F(x), \Phi_G(x), \Phi_G^*(x) \), for each of the corresponding counts \( F_l(n), G_l(n), G_l^*(n) \). Then \( \Phi(x) = \frac{P(x)}{Q(x)} \) where \( Q(x) \) is the co-characteristic polynomial, that is,

\[
Q(x) = x^{l+1}h_l(1/x) = 1 - x - x^{l+1},
\]

and the numerator \( P(x) = Q(x)\Phi(x) \) depends on the respective initial values presented in Theorem 1 and Proposition 3. Thus we get

\[
\Phi_F(x) := \sum_{n \geq 0} F_l(n)x^n = \frac{1 + x + \cdots + x^l}{1 - x - x^{l+1}},
\]

\[
\Phi_G(x) := \sum_{n \geq 0} G_l^*(n)x^n = \frac{l + 1 - lx}{1 - x - x^{l+1}},
\]

\[
\Phi_G(x) := \sum_{n \geq 0} G_l(n)x^n = \Phi_G^*(x) + \sum_{n=2}^l nx^n.
\]

**Remark 4.** In Comtet’s valuable book [4, p. 46] the OGF for the sequence \( G(n, l) \), namely, \((t + (l + 1)t^{l+1})(1 - t - t^{l+1})^{-1} \) which equals \( \Phi_G^*(t) - (l + 1) \), should be replaced by

\[
\Phi_G(t) - l - 1 = (t + (l + 1)t^{l+1})(1 - t - t^{l+1})^{-1} + \sum_{n=2}^l nt^n.
\]

**Proposition 5.** The characteristic roots, roots of \( h_l(z) \), are nonzero and simple.

**Proof.** The constant term of \( h_l(z) \) is nonzero and the only nonzero root of the derivative \( h_l'(z) = (l + 1)z^{l-1}(z - l/(l + 1)) \) does not nullify \( h_l(z) \).

Let \( z_1, z_2, \ldots, z_{l+1} \) be all roots of the characteristic polynomial \( h_l(z) \). Define

\[
\sigma_n(l) = \sum_{j=1}^{l+1} z_j^n,
\]

which is the \( n \)th power sum of characteristic roots.

**Theorem 6.** For integers \( l \geq 0 \) and \( n \geq 1 \), each count \( G_l^*(n) \) of \( l^* \)-independent subsets of the cycle \( \mathbb{Z}_n \) equals the \( n \)th power sum of roots of the characteristic polynomial, i.e., \( G_l^*(n) = \sigma_n(l) \). Additionally, for \( n = 0 \), \( \sigma_0(l) = l + 1 =: G_l^*(0) \).

**Proof.** Let \( P(x) = l + 1 - lx \), \( Q(x) = 1 - x - x^{l+1} \), and let \( t_j, j = 1, \ldots, l + 1 \), be all roots of \( Q(x) \). Hence, by (15), the OGF for \( G_l^*(n) \) is \( \Phi_G^*(x) = P(x)/Q(x) \). Moreover, the reciprocals \( 1/t_j \) are characteristic roots \( z_j \). Due to Proposition 5,
we use the following standard expansion into partial fractions,

\[ \Phi^*_G(x) = \sum_{j=1}^{\frac{l+1}{2}} \frac{P(t_j)}{Q'(t_j)} \cdot \frac{1}{x - t_j} = \frac{1}{\sum_{j=1}^{\frac{l+1}{2}} \frac{P(t_j)}{Q'(t_j)} (1 - xz_j)} \]

\[ = \sum_{n=0}^{\infty} x^n \sum_j c_j \cdot (z_j)^n \]

where

\[ c_j := \frac{P(t_j)}{-t_j Q'(t_j)} = \frac{1 + l \cdot (1 - t_j)}{(t_j + t_j^{l+1}) + t_j^{l+1}} = 1, \quad j = 1, \ldots, l + 1, \]

because \( Q(t_j) = 0 \), i.e., \( t_j^{l+1} = 1 - t_j \) for each root \( t_j \). Thus \( G_l^*(n) = [x^n] \Phi^*_G(x) = \sigma_n(\ell) \), which completes the proof.

**Corollary 7.** The count \( G_l(n) \) of \( l \)-independent subsets of the cycle \( C_n \) is the \( n \)th power sum \( \sigma_n(l) \), i.e., \( G_l(n) = G_l^*(n) \), unless \( l \geq 2 \) and \( 2 \leq n \leq l \).

This corollary gives rise to closed formulas for \( G_l(n) \) if \( l \) is small, \( l \leq 4 \). The formulas are known for \( l = 0, 1 \) and \( n \geq 0 \). Namely,

\[ G_0(n) = 2^n, \quad G_1(n) = \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n = L(n), \quad \text{the} \ n \text{th Lucas number}. \]

For \( l = 2, 3, 4 \) the formulas for roots due to Cardano-del Ferro-Tartaglia (\( l=2,4 \); since \( h_4(z) = (z^3 - z - 1)(z^2 - z + 1) \)) on one hand and Ferrari (\( l = 3 \)) on the other hand and the de Moivre formula are helpful, see the result in [12, formula (11)] for \( G_2(n) \) with \( n > 2 \) only.

6. **Main Result Via Newton’s Formulas**

Given a degree-\( r \) characteristic polynomial \( f(x) = x^r + a_1x^{r-1} + \cdots + a_r \), its \( n \)th moment, \( S_n \), being the \( n \)th power sum of roots of \( f(x) \), satisfies the order-\( r \) recurrence corresponding to \( f(x) \), namely, \( S_n + a_1S_{n-1} + \cdots + a_rS_{n-r} = 0 \) for each \( n \geq r \). It is so because the general solution includes \( S_n \) as a particular solution. Initial values \( S_k \) for \( k = 0, 1, \ldots, r - 1 \) (\( S_0 = r, S_1 = -a_1 \)) can be obtained for \( k \geq 1 \) recursively from the following Newton formulas: \(-na_n = S_n + a_1S_{n-1} + \cdots + a_{n-1}S_1 \) where \( n = 1, 2, \ldots, \) with \( a_k = 0 \) for \( k > \deg f(x) = r \).

**Alternative proof of Theorem 6.** The moment \( \sigma_n(l) \) and the count \( G_l^*(n) \) satisfy the same recurrence with characteristic polynomial \( h_l(z) \) of degree \( r := l + 1 \) and with only two nonzero coefficients \( a_j \), namely \( a_1 = -1 = a_r \). Hence, due to
Newton’s formulas, the \( r \) initial values of \( \sigma_n(l) \), for \( n = 0, 1, \ldots, r - 1 = l \), are \( l + 1, 1, \ldots, 1 \), and these are initial values of \( G^*_l(n) \) due to (9) and (5).

For the case \( l = 2 \) only, a similar proof in [12, Lemma 10 and Remark 3.2] uses the Viète formulas (instead of Newton’s).

7. Asymptotics

The following celebrated result is of basic importance in asymptotic analysis of combinatorial counting sequences, see [5].

Theorem 8 (Pringsheim’s Theorem). Let \( f(z) \) be a power series analytic at the origin \( z = 0 \), with nonnegative coefficients and with finite radius of convergence \( R \). Then the point \( z = R \) is a dominant pole (of least magnitude) of the function \( f(z) \).

A polynomial \( Q(x) \in \mathbb{Z}[x] \) is called a multi-composition polynomial if \( Q(x) = 1 - \sum_{j=1}^{\nu} m_j x^{a_j} \) where all \( \nu \geq 2 \), \( m_j \)'s and \( 1 \leq a_1 < a_2 < \cdots < a_\nu \) are natural numbers of which \( a_i \)’s are relatively prime, \( \gcd\{a_1, \ldots, a_\nu\} = 1 \). Then the co-reciprocal polynomial of \( Q(x) \), say \( h(x) := x^{a_\nu} Q(1/x) \), is the characteristic polynomial of a ‘compositional’ recurrence (for a ‘compositional’ count \( u(n) \)), \( u(n) = \sum_{j=1}^{\nu} m_j u(n - a_j) \), generated by \( Q(x) \) via the above LinHomConstR-OGF. Elementary reasoning gives the following result.

Lemma 9 (Skupień [13]). Any multi-composition polynomial has a simple positive root, \( \tau \), which is smaller than the minimum magnitude among remaining roots, if any, and \( \tau < 1 \).

Corollary 10. If \( u(n) \) is a compositional count with nontrivial natural initial terms and \( \lambda \) is a characteristic root of largest magnitude then \( \lambda \) is a simple positive root, \( \lambda > 1 \), and \( u(n) = \Theta(\lambda^n) \), the exact asymptotic order of growth.

This result applies to our counts due to Theorems 1 and 6, and Corollary 7. Hence,

Proposition 11. If \( \lambda(l) \) stands for the dominant root of the characteristic polynomial \( h_l(z) = z^{l+1} - z^l - 1 \) then \( F_l(n) = \Theta(\lambda(l)^n) \), both \( G^*_l(n) \), \( G_l(n) \sim \lambda(l)^n \), and \( G_l(n) = \lfloor \lambda(l)^n \rfloor \) for \( n \geq 2 \) if \( l = 1 \), \( n \geq 6 \) if \( l = 2 \), and \( n \geq 22 \) if \( l = 3 \).

Remark 12. It can be seen, for \( l \leq 3 \) only, that magnitudes of remaining characteristic roots are less than 1 and therefore nearest integer function is applicable.

Moreover, the initial \( \lambda(l) \)'s are important in the subclass of algebraic integers which comprises Pisot numbers [3, 17]: golden ratio \( (l = 1) \) and next the 4th \( (l = 2) \),
Sums of Powered Characteristic Roots

2nd \((l = 3)\), and 1st \((l = 4)\) of the smallest Pisot numbers, the smallest being called the plastic number, and its minimal polynomial is the degree-3 factor of \(h_4(z)\), \(h_4(z) = (z^3 - z - 1)(z^2 - z + 1)\).

<table>
<thead>
<tr>
<th>(l)</th>
<th>(\lambda(l))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.61803(^+)</td>
</tr>
<tr>
<td>2</td>
<td>1.46557(^+)</td>
</tr>
<tr>
<td>3</td>
<td>1.38028(^-)</td>
</tr>
<tr>
<td>4</td>
<td>1.32472(^-)</td>
</tr>
</tbody>
</table>

Table 1. Pisot numbers.

8. Hamilton Cycles in a Squared Cycle

Investigations into distance-independent circular sets, presented above, have been inspired by the problem of counting Hamilton cycles (i.e., connected 2-factors) in the square of a cycle [11, 12]. Recall that the square of the \(n\)-cycle \(C_n\), in symbols \(C_n^2\), is the graph \(C_n\) together with all \(n\) shortest chords (all chords of length two). One of the main results in [12] is the following closed formula which gives the number, \(h(C_n^2)\), of Hamilton cycles in \(C_n^2\) for \(n \geq 5\) in terms of the number, \(G_2(n) = G_2^*(n)\), of 2-independent sets on the \(n\)-cycle. Namely, if

\[
h_n := G_2^*(n) + 2 \lceil n/2 \rceil,
\]

then \(h(C_n^2) = h_n\) for \(n \geq 5\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_2^*(n))</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
<td>31</td>
<td>46</td>
</tr>
<tr>
<td>(h_n)</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>12</td>
<td>16</td>
<td>23</td>
<td>29</td>
<td>41</td>
</tr>
</tbody>
</table>

Table 2

Values of the extended \(h_n\) such that (18) holds for arguments \(n \geq 0\) are presented in Table 2. Note that the result \(h_n = h(C_n^2)\) does not extend to \(n = 4\) because \(h(C_4^2) = h(K_4) = 3 \neq h_4 = 9\). (In general, \(h(K_n) = \lfloor (n-1)!/2 \rfloor\). That is why \(h_5 = h(K_5) = 12\).)

**Proposition 13.** For the extended sequence \(h_n\), OGF: \(\frac{3-2x}{1-x^2} + \frac{x}{(1-x)^2} + \frac{x}{1-x^2}\), \(h_n = 2h_{n-1} - h_{n-3} - h_{n-5} + h_{n-6}\) for \(n \geq 6\), with initial conditions included in Table 2.

**Proof.** Due to (15) with \(l = 2\), it is easily seen that the above OGF is the sum of three OGFs one each for three summands in \(h_n = G_2^*(n) + n + (1 - (-1)^n)/2\). Therefore l.c.m., say \(Q(x)\), of denominators of the three partial OGFs is the denominator of the above main OGF,
\[ Q(x) = (1 - x - x^3)(1 - x^2)(1 + x) = 1 - 2x + x^3 + x^5 - x^6. \]

Hence the above Procedure OGF-LinHomRec gives the stated recurrence (of order six).

9. Concluding Remarks

Inspired by the above study is the following recent theorem related to very old Girard-Newton-Waring’s formulas for moments (power sums of roots) of a polynomial. The theorem seems to be unpublished yet, and this opinion agrees with comments in the introductory part of [8].

**Theorem 14** [14]. Let \( f(z) \) be a polynomial of degree \( r > 0 \) and with nonzero roots only, whereas \( g(z) \) the reciprocal polynomial of \( f(z) \). Let \( S_n(f) \) and \( S_n(g) \) be the \( n \)th moments of \( f \) and \( g \), resp. Then the OGF for moments of \( f(x) \) is

\[
\frac{rg(z) - zg'(z)}{g(z)} = \sum_{n=0}^{\infty} S_n(f)z^n
\]

and OGF for moments of \( g(x) \) results on interchanging symbols \( f \leftrightarrow g \) on both sides of the formula.

**Procedure RootsPowerSums.**

Input \([h(z)], \) a polynomial with nonzero roots\].

Output \([The sequence of power sums of roots of \( h(z) \), represented by the rational OGF \( \frac{P(z)}{Q(z)} \) or by LinHomRec obtainable by Procedure OGF-LinHomRec, see Section 4]\].

Action

- \( Q(z) := z^{\deg h(z)}h(1/z), \) the co-reciprocal polynomial of \( h(z) \);
- \( P(z) := -z^{\deg h(z)}Q'(z) \mod Q(z) \) so that \( P(0) = \deg h(z) \);

Procedure OGF-LinHomRec;

STOP.

Another byproduct (which is useful when dealing with LinHomConst recurrences) is the notion of mutually reciprocal polynomials.

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