ON MINIMUM \((K_q, k)\) STABLE GRAPHS

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Abstract

A graph \(G\) is a \((K_q, k)\) stable graph \((q \geq 3)\) if it contains a \(K_q\) after deleting any subset of \(k\) vertices \((k \geq 0)\). Andrzej Źak in the paper On \((K_q; k)\)-stable graphs, (doi:/10.1002/jgt.21705) has proved a conjecture of Dudek, Szymański and Zwonek stating that for sufficiently large \(k\) the number of edges of a minimum \((K_q, k)\) stable graph is \((2q - 3)(k + 1)\) and that such a graph is isomorphic to \(sK_{2q-2} + tK_{2q-3}\) where \(s\) and \(t\) are integers such that \(s(q - 1) + t(q - 2) - 1 = k\). We have proved (Fouquet et al. On \((K_q, k)\) stable graphs with small \(k\), Elektron. J. Combin. 19 (2012) #P50) that for \(q \geq 5\) and \(k \leq \frac{q}{2} + 1\) the graph \(K_{q+k}\) is the unique minimum \((K_q, k)\) stable graph. In the present paper we are interested in the \((K_q, \kappa(q))\) stable graphs of minimum size where \(\kappa(q)\) is the maximum value for which for every nonnegative integer \(k < \kappa(q)\) the only \((K_q, k)\) stable graph of minimum size is \(K_{q+k}\) and by determining the exact value of \(\kappa(q)\).

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1. Introduction

For terms not defined here we refer to [1]. As usual, the order of a graph $G$ is the number of its vertices and the size of $G$ is the number of its edges (denoted by $e(G)$). The disjoint union of two graphs $G_1$ and $G_2$ is denoted by $G_1 + G_2$. The union of $p$ mutually disjoint copies of a graph $G$ is denoted by $pG$. For any set $A$ of vertices, we denote by $G[A]$ the subgraph induced by $A$ and by $G - A$ the subgraph induced by $V(G) - A$. If $A = \{v\}$ we write $G - v$ for $G - \{v\}$. For any set $F$ of edges, we denote by $G - F$ the spanning subgraph $(V(G), E(G) - F)$. If $F = \{e\}$ we write $G - e$ instead of $G - \{e\}$. A complete subgraph of order $q$ of $G$ is called a $q$-clique of $G$. The complete graph of order $q$ is denoted by $K_q$.

When a graph $G$ contains a $q$-clique as subgraph, we say “$G$ contains a $K_q$”.

In [6] Horváth and Katona considered the notion of $(H, k)$ edge stable graph\(^2\): given a simple graph $H$, an integer $k$ and a graph $G$ containing $H$ as subgraph, $G$ is an $(H, k)$ edge stable graph whenever the deletion of any set of $k$ edges does not lead to an $H$-free graph. These authors consider $(P_n, k)$ edge stable graphs and prove a conjecture stated in [5] on the minimum size of a $(P_4, k)$ edge stable graph. In [2], Dudek, Szymański and Zwonek investigated the vertex version of this notion and introduced the $(H, k)$ stable graphs.

**Definition 1.1** [2]. Given an integer $k \geq 0$ and a graph $H$ without isolated vertices, a graph $G$ containing a subgraph isomorphic to $H$ is said to be an $(H, k)$ stable graph if, for every subset $S$ of $k$ vertices, $G - S$ contains (a subgraph isomorphic to) $H$.

**Definition 1.2.** A $(H, k)$ stable graph with minimum size is called a minimum $(H, k)$ stable graph. The size of a minimum $(H, k)$ stable graph shall be denoted by $\text{stab}(H, k)$.

Note that if $G$ is an $(H, k)$ stable graph with minimum size then the graph obtained from $G$ by addition or deletion of some isolated vertices is also minimum $(H, k)$ stable. Hence we shall assume that all the graphs considered in the paper have no isolated vertices. It is clear that $H$ is the unique $(H, 0)$ stable graph with minimum size.

In this paper we consider $(K_q, k)$ stable graphs with $q \geq 2$. Since $K_{q+k}$ is $(K_q, k)$ stable, note that a trivial upper bound for $\text{stab}(K_q, k)$ is $\left(\frac{q+k}{2}\right)$. It is an easy exercise to see that $\text{stab}(K_2, k) = k + 1$ and that the matching $(k+1)K_2$ is the unique minimum $(K_2, k)$ stable graph.

Dudek, Szymański and Zwonek have proved in [2] that $\text{stab}(K_3, k) = 3(k+1)$ for $k \geq 0$ and $\text{stab}(K_4, k) = 5(k+1)$ for $k \geq 1$ and they have obtained an upper

\(^2\)In the original paper [6] these graphs are just called $(H, k)$ stable by the authors.
bound for $\text{stab}(K_q,k)$ for sufficiently large $k$. More precisely, they have obtained the following result.

**Theorem 1.3** [2]. For every $q \geq 5$, there exists an integer $k(q)$ such that for every $k \geq k(q)$, $\text{stab}(K_q,k) \leq (2q - 3)(k + 1)$.

In order to obtain Theorem 1.3, the authors consider the graph $G = sK_{2q-2} + (r-s)K_q$ with $q \geq 5$, $k \geq (q-1)(q-2)$, $r \in \{1, \ldots, k+1\}$, $s \in \{0, \ldots, r\}$ and $r(q-2) + s - 1 = k$ and note that the number of edges of $G$ is $(2q - 3)(k + 1)$. A smaller bound for $k(q)$ can be obtained by the following Proposition 1.4 (a consequence of an old result of Sylvester [7]; see a proof at the end of Section 2), and more generally apart from $k \in \{0, \ldots, q - 4\}$, Theorem 1.6 below gives a better upper bound than $(q+k)$ for $\text{stab}(K_q,k)$.

**Proposition 1.4.** Let $q \geq 4$ be an integer. Set

$$A(q) = \bigcup_{0 \leq i \leq q-4} \{ i(q-1) + j \mid 0 \leq j \leq q - 4 - i \}$$

and

$$B(q) = \{ b \in \mathbb{N} \mid 0 \leq b \leq (q-2)(q-3) - 2 \} - A(q).$$

Let $k$ be a nonnegative integer. There exist integers $s$ and $t$ such that $s(q-1) + t(q-2) - 1 = k$ if and only if $k \in B(q)$ or $k \geq k(q) = (q-3)(q-2) - 1$. For such a pair $(s, t)$, $G = sK_{2q-2} + tK_{q-3}$ is $(K_q,k)$ stable and $e(G) = (2q - 3)(k + 1)$.

Note that $|A(q)| = \frac{(q-3)(q-2)}{2}$ and $|B(q)| = |A(q)| - 1$.

**Lemma 1.5.** Let $q \geq 4$ and $k \geq 0$ be integers. Then $k \in A(q)$ if and only if $[\frac{k+1}{q-1}, \frac{k+1}{q-2}]$ contains no integer.

**Theorem 1.6.** Let $q \geq 3$ and $k \geq 0$ be integers. Set $A(3) = B(3) = \emptyset$, and for $q \geq 4$, $A(q)$ and $B(q)$ are the sets defined in Proposition 1.4. For every positive integer $r$ set

$$\phi(r) = \frac{1}{2} \left( q - 1 + \left\lfloor \frac{k+1}{r} \right\rfloor \right) \left( q - 2 - \left\lfloor \frac{k+1}{r} \right\rfloor \right) r + 2(k + 1).$$

Then, $\text{stab}(K_q,k)$ is at most equal to

- $\phi(1) = \frac{1}{2} (q + k - 1)(q + k)$ if $k \leq q - 4$ (note that $k$ is in $A(q)$),
- $\min \{ \phi(\left\lfloor \frac{k+1}{r} \right\rfloor), \phi(\left\lfloor \frac{k+1}{r} \right\rfloor + 1) \}$ if $k \in A(q)$ and $k \geq q - 1$,
- $(2q - 3)(k + 1)$ if $k \in B(q)$ or $k \geq k(q) = (q - 3)(q - 2) - 1$ (note that $\phi(r) = (2q - 3)(k + 1)$ for every integer $r \in \left[ \frac{k+1}{q-1}, \frac{k+1}{q-2} \right]$).

We shall give a proof of Theorem 1.6 in Section 3 by considering $(K_q,k)$ stable graphs having cliques as components and having the minimum number of edges. As a consequence, if every component of a minimum $(K_q,k)$ stable graph is
complete (see Problem 1.15) then the upper bound given in Theorem 1.6 is the exact value for $\text{stab}(K_q, k)$.

In light of their results, Dudek, Szymański and Zwonek propose the following conjecture.

**Conjecture 1.7** [2]. There exists an integer $k(q)$ such that for every $k \geq k(q)$, $\text{stab}(K_q, k) = (2q - 3)(k + 1)$.

Note that Conjecture 1.7 is true for $q \in \{3, 4\}$. In [4] we have proved that $\text{stab}(K_5, k) = 7(k + 1)$ for $k \geq 5$, which confirms Conjecture 1.7 for $q = 5$. Moreover, we have characterized $(K_q, k)$ stable graphs with minimum size for $q \in \{3, 4, 5\}$. The following theorem summarizes these results.

**Theorem 1.8** [4]. Let $G$ be a minimum $(K_q, k)$ stable graph, with $q \in \{3, 4, 5\}$ and $k \geq k(q)$ with $k(3) = 0$, $k(4) = 1$, $k(5) = 5$. Then $G = sK_{2q-2} + tK_{2q-3}$, for any choice of $s$ and $t$ such that $s(q - 1) + t(q - 2) - 1 = k$. Moreover, $K_{5+k}$ is the unique minimum $(K_5, k)$ stable graph for $k \in \{1, 2, 3\}$, $K_9$ and $K_6 + K_7$ are the only minimum $(K_5, 4)$ stable graphs.

An important fact is that Conjecture 1.7 of Dudek, Szymański and Zwonek has been recently solved by Žak [8], who has characterized also the extremal graphs.

**Theorem 1.9** [8]. Let $q \geq 2$, $k \geq 0$ be nonnegative integers. Then $\text{stab}(K_q, k) \geq (2q - 3)(k + 1)$, with equality if and only if $k = s(q - 1) + t(q - 2) - 1$ for some nonnegative integers $s$ and $t$. In particular, $\text{stab}(K_q, k) = (2q - 3)(k + 1)$ for $k \geq (q - 3)(q - 2) - 1$. Furthermore, if $G$ is a $(K_q, k)$ stable graph having exactly $(2q - 3)(k + 1)$ edges, then $G = sK_{2q-2} + tK_{2q-3}$ where $s$ and $t$ are nonnegative integers such that $s(q - 1) + t(q - 2) - 1 = k$.

**Remark 1.10.** Since $(K_q, k)$ stable graphs with minimum size for $q \in \{3, 4, 5, 6\}$ have been characterized (see Theorem 1.8 for $q \leq 5$ and [8] for $q = 6$), to close the study of minimum $(K_q, k)$ stable graphs we have only to consider $q \geq 7$ and $k \in A(q)$ (the set defined in Proposition 1.4).

We have proved in [4] that $K_{q+k}$ is the unique minimum $(K_q, k)$ stable graph for $q \geq 4$ and $k \in \{1, 2\}$, that $K_{q+3}$ is the unique minimum $(K_q, 3)$ stable graph for $q \geq 5$ and in [3] that $K_{q+k}$ is the unique $(K_q, k)$ stable graph for $q \geq 6$ and $k \leq \frac{q}{2} + 1$. Remark that $(\frac{q+k}{2}) - (2q - 3)(k + 1)$ and that this integer is positive for $q \geq 3$ and $k \notin \{q - 3, q - 2\}$. Then, as a consequence of Proposition 1.4, for $q \geq 4$ and for every integer $k$ for which $k \in B(q) - \{q - 3, q - 2\}$ or $k \geq (q - 3)(q - 2) - 1$ the graph $K_{q+k}$ is not minimum $(K_q, k)$ stable. Hence, the set $\{k \in \mathbb{N} \mid K_{q+k}$ is minimum $(K_q, k)$ stable} is bounded above, and we propose the following definition.
Definition 1.11. For every integer \( q \geq 4 \), we denote by \( \kappa(q) \) the greatest integer such that for \( 1 \leq k < \kappa(q) \) the only minimum \( (K_q,k) \) stable graph is \( K_q+k \).

We will focus our attention on determining the exact value of \( \kappa(q) \). In two previous papers we have proved the following.

Theorem 1.12 \([3, 4]\). \( \kappa(3) = 1 \), \( \kappa(4) = 3 \), \( \kappa(5) = 4 \) and for \( q \geq 6 \), \( \kappa(q) > \frac{q}{2} + 1 \).

In this paper we give an upper bound for the value of \( \kappa(q) \).

Theorem 1.13. For every \( q \geq 4 \), if \( \kappa(q) \) is even, then \( \kappa(q) < \sqrt{2(q-1)(q-2)} \) and if \( \kappa(q) \) is odd, then \( \kappa(q) < \sqrt{1 + 2(q-1)(q-2)} \).

We prove that these upper bounds are reached for values of \( q \) such that there exists a minimum \( (K_q, \kappa(q)) \) stable disconnected graph (note that it is the case for \( q = 4 \) and \( q = 5 \)).

Theorem 1.14. Let \( q \geq 4 \) and suppose that there exists a disconnected minimum \( (K_q, \kappa(q)) \) stable graph. Set \( \rho(q) = \left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 1 \).

If \( \frac{1}{2}(q-1)(q-2) > \rho(q)^2 + \rho(q) \), then \( \kappa(q) = 2\rho(q) + 1 \).

If \( \frac{1}{2}(q-1)(q-2) \leq \rho(q)^2 + \rho(q) \), then \( \kappa(q) = 2\rho(q) \).

Proofs of Theorems 1.13 and 1.14 shall be given in Subsection 3.3.

Remark that, by definition of \( \kappa(q) \) and by Theorem 1.9, for every integer \( k \) in \( \{ l \in \mathbb{N} \mid 0 \leq l < \kappa(q) \text{ or } l \geq (q-2)(q-3)-1 \} \cup B(q) \) every component of any minimum \( (K_q, k) \) stable graph is complete, but we do not know if it is true for \( k \) in \( \{ l \in \mathbb{N} \mid l \geq \kappa(q) \text{ and } l \in A(q) \} \) (where \( A(q) \) and \( B(q) \) are the sets defined in Proposition 1.4).

If there is no minimum disconnected \( (K_q, \kappa(q)) \) stable graph then, by definition of \( \kappa(q) \), there exists a connected minimum \( (K_q, \kappa(q)) \) stable graph \( G_q \) which is not complete. We think that it never happens, so we propose the following problem.

Problem 1.15. Is it true that if \( G \) is a minimum \( (K_q, k) \) stable graph, then every component of \( G \) is complete?

If the answer is positive then Theorem 1.14 gives the exact value of \( \kappa(q) \) for every \( q \geq 4 \).

2. General Results

Lemma 2.1 \([2]\). Let \( G \) be an \( (H, k) \) stable graph with \( k \geq 1 \). Then, for every vertex \( v \), \( G - v \) is \( (H, k-1) \) stable.
A set of vertices of $G$ that intersects every subgraph of $G$ isomorphic to $H$ is called a transversal of all the subgraphs isomorphic to $H$ or simply an $H$-transversal of $G$. An $H$-transversal of $G$ having the minimum number of vertices is said to be a minimum $H$-transversal of $G$. The number of vertices of a minimum $H$-transversal is denoted by $\tau_H(G)$. Remark that $G$ is $(H, k)$ stable if and only if $\tau_H(G) \geq k + 1$

**Definition 2.2.** Let $G$ be an $(H, k)$ stable graph. If $G$ has a minimum $H$-transversal having exactly $k + 1$ vertices, $G$ is said to be exactly $(H, k)$ stable.

**Lemma 2.3** [2]. Let $G$ be an $(H, k)$ stable graph with $k \geq 1$ and $e \in E(G)$ such that $G - e$ is not $(H, k)$ stable. Then $G$ is exactly $(H, k)$ stable and $G - e$ is exactly $(H, k - 1)$ stable.

**Definition 2.4** [2]. Let $G$ be an $(H, k)$ stable graph. If $G - e$ is not $(H, k)$ stable for every edge $e \in E(G)$, then $G$ is said to be minimal $(H, k)$ stable.

**Remark 2.5.** In [2] “minimal $(H, k)$ stable graphs” are called “strong $(H, k)$ stable graphs” by the authors. Note that an $(H, k)$ stable graph $G$ is minimal $(H, k)$ stable if and only if for every $e \in E(G)$ the graph $G - e$ is exactly $(H, k - 1)$ stable. Moreover, a minimal $(H, k)$ stable graph is exactly $(H, k)$ stable.

If there exists an edge $e$ of an $(H, k)$ stable graph $G$ such that there are no subgraphs isomorphic to $H$ containing $e$, then $G - e$ is an $(H, k)$ stable graph. Hence, we have the following.

**Lemma 2.6** [2]. Every edge of a minimal $(H, k)$ stable graph is contained in a subgraph isomorphic to $H$. Consequently, every vertex of a minimal $(H, k)$ stable graph is also contained in a subgraph isomorphic to $H$.

**Remark 2.7.** Clearly, every minimum $(H, k)$ stable graph is minimal $(H, k)$ stable.

One may ask what happens for components of an $(H, k)$ stable graph. The following theorem gives us an answer when $H$ is connected. We shall say that a graph containing no subgraph isomorphic to $H$ is $(H, -1)$ stable.

**Theorem 2.8.** Let $H$ be a connected graph containing at least 2 vertices, let $G$ be an exactly $(H, k)$ stable graph, and let $G_1$, $G_2$, ..., $G_r$, with $r \geq 1$, be its components. Then, there exist integers $k_1$, $k_2$, ..., $k_r$, with $0 \leq k_i \leq k$, such that

(i) for every $i$, with $1 \leq i \leq r$, $G_i$ is exactly $(H, k_i)$ stable,

(ii) $\sum_{i=1}^r k_i + (r - 1) = k$,

$G$ is minimal $(H, k)$ stable if and only if for every $i$, $1 \leq i \leq r$, $G_i$ is minimal $(H, k_i)$ stable. Moreover, if $G$ is minimum $(H, k)$ stable, then for every $i$, $1 \leq i \leq r$, $G_i$ is minimum $(H, k_i)$ stable.
For each $i$, $1 \leq i \leq r$, let us consider a minimum $H$-transversal of $G_i$, say $T_i$, and set $k_i = |T_i| - 1$. Clearly, for each $i$ the graph $G_i$ is exactly $(H, k_i)$ stable and the set $T = \bigcup_{1 \leq i \leq r} T_i$ is a minimum $H$-transversal of $G$. Note that the number of elements of $T$ is $|T| = \sum_{i=1}^{r} k_i + r$ and we have $|T| > k$. Let $S$ be any set of vertices of $G$ such that $|S| \leq |T| - 1$ and for every $i$ denote by $S_i$ the set $S \cap V(G_i)$. Clearly, there exists $i_0 \in \{1, \ldots, r\}$ such that $|S_{i_0}| \leq k_{i_0} = |T_{i_0}| - 1$. Then, $G_{i_0} - S_{i_0}$ contains a subgraph isomorphic to $H$, that is, $G$ is exactly $(H, |T| - 1)$ stable, and we have $\sum_{i=1}^{r} k_i + (r - 1) = k$.

Let $e$ be an edge of $G$ and let $G_i$ be the component containing $e$.

Claim. $G - e$ is $(H, k)$ stable if and only if $G_i - e$ is $(H, k_i)$ stable.

**Proof.** Suppose that $G_i - e$ is $(H, k_i)$ stable. Let $U$ be an $H$-transversal of $G - e$. Set $U_i = U \cap V(G_i - e) = U \cap V(G_i)$ and for every $j \neq i$, $U_j = U \cap V(G_j)$. Since $(G_i - e) - U_i$ and each $G_j - U_j$, $j \neq i$, contain no subgraphs of $G - e$ isomorphic to $H$, we have for every $j$, $1 \leq j \leq r$, $|U_j| \geq k_j + 1$. Then, $|U| = \sum_{j=1}^{r} |U_j| \geq k + 1$. Hence, for every set $S$ of $k$ vertices $(G - e) - S$ contains a subgraph isomorphic to $H$, that is, $G - e$ is $(H, k)$ stable.

Conversely, suppose that $G_i - e$ is not $(H, k_i)$ stable. Let $T_i$ be an $H$-transversal of $(G_i - e) - T_i$ having $k_i$ vertices. For every $j \neq i$ let $T_j$ be an $H$-transversal of $G_j$ having $k_j + 1$ vertices. The set $T = \bigcup_{j=1}^{r} T_j$ has $k$ vertices and is a $H$-transversal of $G - e$, that is, $G - e$ is not $(H, k)$ stable.

Thus, $G$ is minimal $(H, k)$ stable if and only if for every $i$, $1 \leq i \leq r$, $G_i$ is minimal $(H, k_i)$ stable.

Note that, by replacing a minimal $(H, k_i)$ stable component $G_i$ by any minimal $(H, k_i)$ stable graph $G_i'$ (connected or not), we obtain again a minimal $(H, k)$ stable graph. Thus, if $G$ is minimum $(H, k)$ stable then for every $i$, $1 \leq i \leq r$, $G_i$ is minimum $(H, k_i)$ stable.

**Remark 2.9.** Let $r \geq 2$ be an integer, $k_1, \ldots, k_r$ be $r$ non-negative integers and $k = \sum_{i=1}^{r} k_i + (r - 1)$. If for every $i$, $1 \leq i \leq r$, $G_i$ is a minimum $(H, k_i)$ stable graph then the disjoint union $G_1 + G_2 + \cdots + G_r$ may not be a minimum $(H, k)$ stable graph. For example, $K_q$ is minimum $(K_q, 0)$ stable, $2K_q$ and $K_{q+1}$ are minimal $(K_q, 1)$ stable, but for $q \geq 4$ since $e(2K_q) > e(K_{q+1})$, the graph $2K_q$ is not minimum $(K_q, 1)$ stable.

Given relatively prime positive integers $a_1, \ldots, a_n$, let us consider the integers that can be expressed as a sum $k_1a_1 + k_2a_2 + \cdots + k_na_n$, where $k_1, k_2, \ldots, k_n$ are nonnegative integers. Any such integer is said to be *representable*. Recall that the *Frobenius Problem* is the following: find the largest non-representable integer (called the *Frobenius number* and denoted by $g(a_1, \ldots, a_n)$). If $n = 2$, the Frobenius number is given by the formula $g(a_1, a_2) = a_1a_2 - a_1 - a_2$. This formula
was discovered by Sylvester in 1884 [7], who also demonstrated that there are a total of \( N(a_1, a_2) = \frac{(a_1-1)(a_2-1)}{2} \) non-representable integers. For the particular case \( a_2 = a_1 - 1 \) one obtains explicitly the set of non-representable integers.

**Lemma 2.10** [7]. Let \( a \geq 3 \) be an integer and the function \( \alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that \( \alpha(s, t) = sa + t(a-1) \). Set \( A = \bigcup_{0 \leq i \leq a-3} \{ ia + j \mid 1 \leq j \leq a - 2 - i \} \). Every \( b \in \mathbb{N} - A \) is representable (that is, there exists a pair \( \{ s, t \} \) of nonnegative integers such that \( b = sa + t(a-1) \)), and every \( b \) in \( A \) is not representable. Moreover, every representable \( b \) has a unique representation \( sa + t(a-1) \) such that \( 0 \leq t \leq a - 1 \).

We shall give a proof of Lemma 2.10 for completeness.

**Proof of Lemma 2.10.** Note that \( \max(A) = (a-1)(a-2) - 1 \), \( |A| = \frac{(a-1)(a-2)}{2} \) and for \( s \geq 0 \) and \( t \geq 1 \), \( \alpha(s, t) = \alpha(s + 1, t - 1) - 1 \).

Consider the infinite matrix \( \{ \alpha(s, t) \}_{s \geq 0, t \geq 0} \). For any \( t \geq 0 \) the values of the diagonal \( \{ \alpha(i, t-i) \mid 0 \leq i \leq t \} \) are the consecutive integers \( \{ t(a-1) + i \mid 0 \leq i \leq t \} \). For \( s \geq 0 \), the values of the (partial) diagonal \( \{ \alpha(s+i, a-i-1) \mid 0 \leq i \leq a-1 \} \) are the consecutive integers \( sa + (a-1)^2, sa + (a-1)^2 + 1, \ldots, sa + a(a-1) \).

Since \( \alpha(0, a-1) = \alpha(a-2, 0) + 1 \) and for every \( s \geq 0 \), \( \alpha(s+a-1, 0) + 1 = \alpha(s+1, a-1) = sa + a(a-1) + 1 \), every integer \( b \geq (a-2)(a-1) \) appears in

\[
\{ \alpha(i, a-2-i) \mid 0 \leq i \leq a-2 \} \cup \bigcup_{s \geq 0} \{ \alpha(s+i, a-i-1) \mid 0 \leq i \leq a-1 \}.
\]

Let \( B = \bigcup_{0 \leq i \leq a-3} \{ \alpha(j, i-j) \mid 0 \leq j \leq i \} = \bigcup_{0 \leq i \leq a-3} \{ i(a-1) + j \mid 0 \leq j \leq i \} \). Clearly \( |B| = |A| \). It is easy to check that \( A \) and \( B \) are disjoint sets and that \( A \cup B = \{ 0, 1, \ldots, (a-2)(a-1) - 1 \} \). Thus, every \( b \in A \) is not representable and for every integer \( b \in \mathbb{N} - A \) there exists a unique pair \( (s, t) \) with \( s \geq 0 \) and \( 0 \leq t \leq a-1 \) such that \( b = sa + t(a-1) \).

**Remark 2.11.** It is easy to see that every representable \( b < a(a-1) \) has a unique representation. For a representable \( b \geq a(a-1) \), since we can choose values of \( t \geq a \), it is possible that \( b = \alpha(s, t) = \alpha(s', t') \) for distinct pairs \( (s, t) \) and \( (s', t') \). Indeed, if \( s \geq a - 1 \), then for every positive integer \( r \leq \lfloor \frac{s}{a-1} \rfloor \), \( \alpha(s, t) = \alpha(s - r(a-1), ra + t) \).

**Proof of Proposition 1.4.** Let us apply Lemma 2.10 to \( a = q-1 \) and \( b = k+1 \). \( B(q) \) is the set of integers \( k \leq (q-3)(q-2) - 3 \) such that \( k+1 \) is representable as \( s(q-1) + t(q-2) \). More precisely, \( B(q) = \bigcup_{1 \leq i \leq q-4} \{ i(q-2) + j - 1 \mid 0 \leq j \leq i \} \).

It is easy to see that the set of integers \( k \) such that \( k+1 \) is not representable as \( s(q-1) + t(q-2) \) is \( A(q) = \bigcup_{0 \leq i \leq q-4} \{ i(q-1) + j \mid 0 \leq j \leq q-4-i \} \).

A minimum \( K_q \)-transversal of \( G = sK_{2q-2} + tK_{2q-3} \) contains exactly \( s(q-1) + t(q-2) = k+1 \) vertices, that is \( G \) is \((K_q, k)\) stable, and it is easy to check that \( e(G) = (2q-3)(k+1) \).
Proof of Lemma 1.5. If there exist integers $s$ and $t$ such that $s(q-1)+t(q-2)=k+1$ then $\frac{k+1}{q-1}=s+t-\frac{t}{q-1}$ and $\frac{k+1}{q-2}=s+t+\frac{s}{q-2}$, and hence $r=s+t\in[k+1,q-1,\frac{k+1}{q-2}]$. Conversely, let $r\in[k+1,q-1,\frac{k+1}{q-2}]$. Then $q-2=\frac{k+1}{r}\leq q-1$. If $k+1=r(q-1)$ then we are done. If $\frac{k+1}{q-1}<r$ then $q-2=\lfloor\frac{k+1}{r}\rfloor$ is the quotient in the division of $k+1$ by $r$. Hence, if $s$ denotes the remainder, then $k+1=r(q-2)+s=s(q-1)+(r-s)(q-2)$. We conclude by applying Proposition 1.4.

3. Minimum $(K_q,k)$ Stable Graphs

In this section we are interested in $(K_q,k)$ stable graphs with minimum size $(q \geq 3)$. Recall that $\text{stab}(K_q,k)=\min\{e(G) \mid G \text{ is } (K_q,k) \text{ stable}\}$.

3.1. Some known results

We give here some known results about this topic.

By Remark 2.5 and Lemma 2.6 we have:

Properties 3.1 [2]. A minimal $(K_q,k)$ stable graphs $G$ has the following properties:

(P1) $G$ is exactly $(K_q,k)$ stable.

(P2) For every edge $e$, $G-e$ is exactly $(K_q,k-1)$ stable.

(P3) For every vertex $v$, $G-v$ is exactly $(K_q,k-1)$ stable.

(P4) Every vertex of $G$ belongs to some $q$-clique of $G$.

(P5) Every edge of $G$ belongs to some $q$-clique of $G$.

Remark 3.2. For any two integers $q \geq 3$ and $k \geq 1$, $K_{q+k}$ is minimal $(K_q,k)$ stable.

Proposition 3.3 [4]. $K_5$ is the unique minimum $(K_4,1)$ stable graph, $K_6$ is the unique minimum $(K_4,2)$ stable graph and for every integer $q \geq 5$ and every integer $k \in \{1,2,3\}$, $K_{q+k}$ is the unique minimum $(K_q,k)$ stable graph.

Dudek et al. [2] defined the family $A_r^{(K_q,k)}$ with $k \geq 0$, $q \geq 3$, $1 \leq r \leq k+1$ as the family of graphs consisting of $r$ complete graphs $K_{i_j}$ with $i_1 \geq \cdots \geq i_r \geq q$ satisfying the condition $\sum_{i=1}^{r}(i_j - q) + (r-1) = k$ and they proved that every graph in $A_r^{(K_q,k)}$ is minimal $(K_q,k)$ stable. We observe that if a $(K_q,k)$ stable graph $G$ is a disjoint union of $r \geq 1$ cliques $K_{i_j}$, $1 \leq j \leq r$, then by Theorem 2.8, $G \in A_r^{(K_q,k)}$. They defined a graph $G \in A_r^{(K_q,k)}$ as a balanced union if $|i_j - i_l| \in \{0,1\}$ for every $j$ and $l$ in $\{1,2,\ldots,r\}$ and they proved that given $q$, 

On Minimum $(K_q,k)$ Stable Graphs
Let \( f \) and \( r \) there is exactly one balanced union \( B_r^{(K_q,k)} \) in \( A_r^{(K_q,k)} \), and that \( B_r^{(K_q,k)} \) has the minimum number of edges among the graphs in \( A_r^{(K_q,k)} \).

In [2] the following lemma has been given. We give its proof for completeness.

**Lemma 3.4** [2]. Let \( G_0 \) be a \((K_q,k_0)\) stable graph \((k_0 \geq 0)\) which has the minimum size among all graphs being a disjoint union of \( r \) cliques \((r \geq 1)\), \( G_j \equiv K_{q+k_j} \) with \( 1 \leq j \leq r, k_j \geq 0 \). There exist nonnegative integers \( s \) and \( k \) such that \( 0 \leq s \leq r - 1, G_0 = sK_{q+k+1} + (r-s)K_{q+k} \) with \( r(k+1) + s = k_0 + 1 \) and \( e(G_0) = \frac{1}{2r}(r(q-1) + k_0 + 1 - s)(r(q-2) + k_0 + 1 + s) \).

**Proof.** Suppose, without loss of generality, that \( k_1 \geq k_2 \geq \cdots \geq k_r \) and that there exist two components \( G_i \) and \( G_j \) with \( i < j \) such that \( k_i - k_j = 2 \). By substituting \( G_i' \equiv K_{q+k_i-1} \) for \( G_i \) and \( G_j' \equiv K_{q+k_j+1} \) for \( G_j \), we obtain a new \((K_q,k)\) stable graph \( G_0' \) such that \( e(G_0') = e(G_0) - (k_i - k_j - 1) < e(G_0) \), which is a contradiction. Thus, for any \( i \) and any \( j \), \( 0 \leq |k_i - k_j| \leq 1 \). Hence, either for any \( i \) and any \( j \), \( k_i \) and \( k_j \) have the same value \( k \) and we have \( G_0 = rK_{q+k} \) with \( k \geq 0 \), or there exist distinct \( k_i \) and \( k_j \) and we have \( G_0 = sK_{q+k+1} + (r-s)K_{q+k} \) with \( k \geq 0 \) and \( 0 \leq s \leq r - 1 \). Hence, a minimum \( K_q\)-transversal of \( G_0 \) has \( k_0 + 1 = s(k+2) + (r-s)(k+1) = s + r(k+1) \) vertices. Note that \( r \) divides \( k_0 + 1 - s \). We have \( 2e(G_0) = s(q+k+1)q(q+k) + (r-s)(q+k)(q+k-1) \). Since \( k+1 = \frac{k_0+1-s}{r} \), we obtain \( e(G_0) = \frac{1}{2r}(r(q-1) + k_0 + 1 - s)(r(q-2) + k_0 + 1 + s) \).

**Remark 3.5.** In Lemma 3.4 the integers \( q, k_0 \) and \( r \) are given. Given \( q \) and \( k_0 \), in order to obtain an upper bound for \( \text{stab}(K_q,k_0) \) we will determine the values of \( r \) for which \( e(G_0(r)) = \frac{1}{2r}(r(q-1) + k_0 + 1 - s)(r(q-2) + k_0 + 1 + s) \) is minimum. We note that if every component of a minimum \((K_q,k_0)\) stable graph is complete then the minimum value of \( e(G_0(r)) \) is exactly \( \text{stab}(K_q,k_0) \).

### 3.2. Proof of Theorem 1.6

First we give a technical lemma used for proving Theorem 1.6.

**Lemma 3.6.** Let \( a \) and \( b \) be positive integers and for \( x > 0 \) consider the real-to-real function

\[
f(x) = \frac{1}{2} \left( a + 1 + \left\lfloor \frac{b}{x} \right\rfloor \right) \left( \left( a - \left\lfloor \frac{b}{x} \right\rfloor \right) x + 2b \right).
\]

Then, \( f \) is continuous on \((0, +\infty)\), nonincreasing on \((0, \frac{b}{a+1}]\), constant on \( [\frac{b}{a+1}, \frac{b}{a}]\) and nondecreasing on \([\frac{b}{a}, +\infty)\). Moreover \( \min \{ f(r) \mid r \in \mathbb{N} \setminus \{0\} \} \) is equal to

- \( f(1) = \frac{1}{2}(a + b + 1)(a + b) \) if \( \frac{b}{a+1}, \frac{b}{a} \) contains no integer and \( b < a \),
- \( \min \{ f(\lfloor \frac{b}{a+1} \rfloor), f(\lfloor \frac{b}{a+1} \rfloor + 1) \} \) if \( \frac{b}{a+1}, \frac{b}{a} \) contains no integer and \( b > a + 1 \),
For every integer \( p \geq 1 \) and every \( x \in \left[ \frac{b}{a+1}, \frac{b}{a} \right] \) we have \( \left\lfloor \frac{b}{x} \right\rfloor = p \), and hence \( f(x) = \frac{1}{2} \left( a + 1 + p \right) (a - p) x + 2b \). It is easy to see that the function \( f \) is continuous on \((0, +\infty)\), nonincreasing on \([0, \frac{b}{a+1}]\), constant on \([\frac{b}{a+1}, \frac{b}{a}]\) and nondecreasing on \([\frac{b}{a}, +\infty)\). The minimum value for \( f(x) \) (with \( x \) a positive real number) is the integer \((2a + 1)b\) and is reached for every real number \( x \) in \([\frac{b}{a+1}, \frac{b}{a}]\). We note that if \( r \) is a positive integer, then \( f(r) \) is a positive integer.

Now we will find the minimum value of \( f(r) \) when \( r \) is a positive integer.

Case 1. \( \left[ \frac{b}{a+1}, \frac{b}{a} \right] \cap \mathbb{N} = \emptyset \). Note that \( 0 < \frac{b}{a} - \frac{b}{a+1} < 1 \) (that is, \( 0 < b < a(a + 1) \)), \( 0 \leq \left\lfloor \frac{b}{a+1} \right\rfloor \leq a \) and \( \left\lfloor \frac{b}{a+1} \right\rfloor < \frac{b}{a+1} < \frac{b}{a} \). The minimum value for \( f(x) \) is a positive integer, \( \left\lfloor \frac{b}{a+1} \right\rfloor + 1 \) such that \( f(1) \leq f(x) \leq f\left( \left\lfloor \frac{b}{a+1} \right\rfloor + 1 \right) \).

Let \( \beta \) be the remainder of the division of \( b \) by \( a + 1 \). In order to obtain the value \( f(\left\lfloor \frac{b}{a+1} \right\rfloor) \) we must know the integer \( p_1 \geq a + 1 \) such that \( \frac{b}{p_1+1} < \left\lfloor \frac{b}{a+1} \right\rfloor \leq \frac{b}{p_1} \).

Since \( \left\lfloor \frac{b}{a+1} \right\rfloor = \frac{b - \beta}{a+1} \), we have \( p_1 = \left\lfloor \frac{b(a+1)}{b-\beta} \right\rfloor \), and hence

\[
f \left( \left\lfloor \frac{b}{a+1} \right\rfloor \right) = \frac{1}{2} (a + 1 + p_1) (a - p_1) \left( \frac{b - \beta}{a+1} \right) + 2b.
\]

In the same way we obtain

\[
f \left( \left\lfloor \frac{b}{a+1} \right\rfloor + 1 \right) = \frac{1}{2} (a + 1 + p_2) \left( (a - p_2) \left( \frac{b + a + 1 - \beta}{a+1} \right) + 2b \right)
\]

with \( p_2 = \left\lfloor \frac{b(a+1)}{b+a+1-\beta} \right\rfloor \).

Case 2. \( \left[ \frac{b}{a+1}, \frac{b}{a} \right] \cap \mathbb{N} \neq \emptyset \). For any integer \( r \) such that \( \frac{b}{a+1} \leq r \leq \frac{b}{a} \), \( f(r) \) is equal to the minimum value \((2a + 1)b\). \( \blacksquare \)

Proof of Theorem 1.6. In order to avoid confusion between “\( k \)” of the statement of Theorem 1.6 and “\( k \)” appearing in the proof of Lemma 3.4, let us replace “\( k \)” by “\( k_0 \)” in the statement of Theorem 1.6. Consider the \((K_q, k_0)\) stable graph \( G_0 \) defined in Lemma 3.4 and see Remark 3.5. We have \( G_0 = sK_{q+k+1} + (r-s)K_{q+k} \) with \( r(k+1) + s = k_0 + 1 \) and \( e(G_0) = \frac{1}{2} (r(q-1) + k_0 + 1 - s)(r(q-2) + k_0 + 1 + s) \). Since \( k + 1 \) is the quotient of \( k_0 + 1 \) divided by \( r \)
and $s$ is the remainder, we have $s = k_0 + 1 - r\lceil \frac{k_0+1}{r} \rceil$. Hence,

$$e(G_0(r)) = \frac{1}{2} \left( q - 1 + \left\lfloor \frac{k_0+1}{r} \right\rfloor \right) \left( \left( q - 2 - \left\lfloor \frac{k_0+1}{r} \right\rfloor \right) r + 2(k_0 + 1) \right).$$

Set $a = q - 2$, $b = k_0 + 1$ and apply Lemma 3.6 and Lemma 1.5.

3.3. Minimum $(K_q, k)$ stable graph for small $k$

In the following, if no confusion is possible, we simply denote the integer $\kappa(q)$ by $\kappa$.

Lemma 3.7. Suppose that $q \geq 4$. If $\kappa$ is even, then $\text{stab}(K_q, \kappa - 1) < e(2K_{q+\frac{q}{2}-1})$ and $\text{stab}(K_q, \kappa) \leq e(2K_{q+\frac{q}{2}} + K_{q+\frac{q}{2}-1})$.

If $\kappa$ is odd, then $\text{stab}(K_q, \kappa - 1) < e(K_{q+\frac{q+1}{2}} + K_{q+\frac{q-3}{2}})$ and $\text{stab}(K_q, \kappa) \leq e(2K_{q+\frac{q-1}{2}})$.

Proof. Recall that, by definition of $\kappa$, $K_{q+\kappa-1}$ is the only minimum $(K_q, \kappa - 1)$ stable. If $\kappa$ is even then $2K_{q+\frac{q}{2}-1}$ is exactly $(K_q, \kappa - 1)$ stable and $K_{q+\frac{q}{2}} + K_{q+\frac{q}{2}-1}$ is exactly $(K_q, \kappa)$ stable. If $\kappa$ is odd then $K_{q+\frac{q+1}{2}} + K_{q+\frac{q-3}{2}}$ is exactly $(K_q, \kappa - 1)$ stable and $2K_{q+\frac{q-1}{2}}$ is exactly $(K_q, \kappa)$ stable.

Lemma 3.8. Let $q \geq 3$ and $p \geq 0$ be two integers. Then,

$e(K_{q+2p}) < e(K_{q+p} + K_{q+p-1})$ if and only if $p^2 + p < \frac{1}{2}(q-1)(q-2)$ and

$e(K_{q+2p}) = e(K_{q+p} + K_{q+p-1})$ if and only if $p_0 = \frac{1}{2}(\sqrt{1 + 2(q-1)(q-2)} - 1)$ is an integer and $p = p_0$.

$e(K_{q+2p+1}) < e(2K_{q+p})$ if and only if $(p+1)^2 < \frac{1}{2}(q-1)(q-2)$ and

$e(K_{q+2p+1}) = e(2K_{q+p})$ if and only if $p_1 = \frac{1}{2}(\sqrt{2(q-1)(q-2)} - 1)$ is an integer and $p = p_1$.

Proof. It is easy to check that $e(K_{q+2p}) - e(K_{q+p} + K_{q+p-1}) = p^2 + p - \frac{1}{2}(q-1)(q-2)$ and $e(K_{q+2p+1}) - e(2K_{q+p}) = (p+1)^2 - \frac{1}{2}(q-1)(q-2)$. These polynomials of degree 2 in $p$ have positive roots $p_0 = \frac{1}{2}(\sqrt{1 + 2(q-1)(q-2)} - 1)$ and $p_1 = \frac{1}{2}(\sqrt{2(q-1)(q-2)} - 1)$ respectively.

Proof of Theorem 1.13. If $\kappa = 2p$ then, by Lemma 3.7, $\text{stab}(K_q, \kappa - 1) < e(2K_{q+\frac{q}{2}-1})$. Since $\kappa - 1 = 2(p-1) + 1$, by Lemma 3.8, $p^2 < \frac{1}{2}(q-1)(q-2)$, that is, $\kappa < \sqrt{2(q-1)(q-2)}$.

If $\kappa = 2p + 1$ then by Lemma 3.7, $\text{stab}(K_q, \kappa - 1) < e(K_{q+\frac{q+1}{2}} + K_{q+\frac{q-3}{2}})$. Since $\kappa - 1 = 2p$, by Lemma 3.8, $p < \frac{1}{2}(\sqrt{1 + 2(q-1)(q-2)} - 1)$, that is, $\kappa < \sqrt{1 + 2(q-1)(q-2)}$. 

Theorem 3.9. Let \( q \geq 4 \) and suppose that there exists a minimum \((K_q, \kappa)\) stable graph \( G_0 \) which is disconnected. Then \( G_0 \) is isomorphic to \( K_q + \lfloor \frac{\kappa}{2} \rfloor + K_q + \lceil \frac{\kappa}{2} \rceil \).

Proof. Let \( G_0 \) be a minimum \((K_q, \kappa)\) stable disconnected graph having \( r \geq 2 \) connected components \( G_1, G_2, \ldots, G_r \). By Theorem 2.9, there are integers \( k_1 \geq k_2 \geq \cdots \geq k_r \) with \( \sum_{i=1}^r k_i + (r - 1) = \kappa \) such that for \( 1 \leq i \leq r \), \( G_i \) is minimum \((K_{k_i}, k_i)\) stable. For every \( i \), since \( k_i < \kappa \), we have \( G_i \cong K_{k_i} \).

Let us suppose that \( r \geq 3 \). We have \( k_r + k_{r-1} = \kappa - (k_{r-2} + k_{r-3} + \cdots + k_1) - (r - 1) \leq \kappa - 2 \). Hence, \( e(K_{k_r+k_{r-1}}) < e(K_{k_r+k_{r-1}}) + e(K_{k_{r-2}+k_{r-3}}) \). The graph \( K_{k_r+k_{r-1}} + K_{k_{r-2}+k_{r-3}} + \cdots + K_{k_1} \) is \((K_q, \kappa)\) stable with strictly smaller size than \( K_{k_1} + K_{k_2} + \cdots + K_{k_r} \), a contradiction. Hence, \( r = 2 \), \( G_0 \in B_2(K_q^{(n)}) \) and by Lemma 3.4 the theorem follows.

Note that Theorem 3.9 implies that there exists at most one disconnected minimum \((K_q, \kappa)\) stable graph and this graph, if it exists, is

- either isomorphic to \( K_q + \frac{\kappa}{2} + K_q + \frac{\kappa}{2} - 1 \) (if \( \kappa \) is even)
- or else isomorphic to \( 2K_q + \frac{\kappa}{2} - 1 \) (if \( \kappa \) is odd).

Proof of Theorem 1.14. By Lemma 3.7 and Theorem 3.9,

if \( \kappa \) is odd, then

\[
e(K_{q+\kappa-1}) < e(K_{q+\frac{\kappa}{2}-1} + K_q + \frac{\kappa}{2}) < stab(K_q, \kappa) = e(2K_q + \frac{\kappa}{2}) \leq e(K_q + \kappa)
\]

(note that, by Lemma 3.8, it may be possible that \( e(2K_q + \frac{\kappa}{2}) = e(K_q + \kappa) \) for some values of \( q \));

if \( \kappa \) is even, then

\[
e(K_{q+\kappa-1}) < e(2K_q + \frac{\kappa}{2} - 1) < stab(K_q, \kappa) = e(K_q + \frac{\kappa}{2} + K_q + \frac{\kappa}{2} - 1) \leq e(K_q + \kappa)
\]

(note that, by Lemma 3.8, it may be possible that \( e(K_q + \frac{\kappa}{2} + K_q + \frac{\kappa}{2} - 1) = e(K_q + \kappa) \) for some values of \( q \)).

For \( \kappa = 2p + 1 \) we have

\[
\frac{1}{2}(q + 2p)(q + 2p - 1) < (q + p - 1)^2 < (q + p)(q + p - 1) \leq \frac{1}{2}(q + 2p + 1)(q + 2p).
\]

This implies that

\[
(A) \quad p^2 + p < \frac{1}{2}(q - 1)(q - 2) \leq (p + 1)^2.
\]

For \( \kappa = 2p \) we have

\[
\frac{1}{2}(q + 2p - 1)(q + 2p - 2) < (q + p - 1)(q + p - 2) < (q + p - 1)^2 \leq \frac{1}{2}(q + 2p)(q + 2p - 1).
\]
This implies that
\[(B) \quad p^2 < \frac{1}{2}(q-1)(q-2) \leq p^2 + p.\]
Combining (A) and (B) yields
\[p^2 < \frac{1}{2}(q-1)(q-2) \leq (p+1)^2.\]
This implies that
\[\sqrt{\frac{1}{2}(q-1)(q-2)} - 1 \leq p < \sqrt{\frac{1}{2}(q-1)(q-2)}.\]
Hence,
\[p = \rho(q) = \left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 1.\]
By inequalities (A) and (B), position of $\frac{1}{2}(q-1)(q-2)$ in comparison to $\rho(q)^2 + \rho(q)$ determines the parity of $\kappa$. Hence, if $\frac{1}{2}(q-1)(q-2) > \rho(q)^2 + \rho(q)$, then $\kappa = 2\rho(q) + 1 = 2 \left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 1$ else $\kappa = 2\rho(q) = 2 \left\lceil \sqrt{\frac{1}{2}(q-1)(q-2)} \right\rceil - 2$.

If there is no minimum disconnected $(K_q, \kappa(q))$ stable graph then, by definition of $\kappa(q)$, there exists a connected minimum $(K_q, \kappa(q))$ stable graph $G_q$ distinct from a clique. Note that if such a graph exists, then
\[e(G_q) < \min\{e(K_q+\kappa), e(K_{q+\frac{\kappa}{2}} + K_{q+\frac{\kappa}{2} - 1})\}, \text{ if } \kappa = \kappa(q) \text{ is even}\]
or
\[e(G_q) < \min\{e(K_q+\kappa), e(2K_{q+\frac{\kappa}{2} - 1})\}, \text{ if } \kappa = \kappa(q) \text{ is odd}.\]
A positive answer to Problem 1.15 states that there is no such graph $G_q$.

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