NOTE ON THE CORE MATRIX PARTIAL ORDERING

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Abstract

Complementing the work of Baksalary and Trenkler [2], we announce some results characterizing the core matrix partial ordering.

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1. Preliminaries

Let $\mathbb{C}^{m \times n}$ be the set of $m \times n$ matrices with complex entries. We will denote the conjugate transpose, range (column space), and nullspace of $A \in \mathbb{C}^{m \times n}$ by $A^*$, $R(A)$, and $N(A)$, respectively. $P_A$ will stand for the orthogonal projector on $R(A)$. We use $I$ to denote an identity matrix with dimensions following from the context.

We start by stating several basic facts on generalized inverses. As references, one can consult [4, Sections 2.2–2.5] or [5, Sections 4.2–4.5].

We let $A^{-}$ designate a generalized inverse of $A$, this being defined as a solution to the matrix equation $AXA = A$. A least squares generalized inverse of $A \in \mathbb{C}^{m \times n}$, written as $A^{-}_L$, is defined to be a solution to the matrix equation $AX = P_A$ ([4, Theorem 2.5.14]). The collection of all $A^{-}_L$ is denoted by $\{A^{-}_L\}$. In light of Theorems 2.5.24 (ii) and 2.5.27 in [4], the
Moore-Penrose inverse of $A$ is the unique element $A^+$ of $\{ A^-_\ell \}$ with the property $R(A^+) = R(A^*)$. The general expression of $A^+$ can be written as $A^+ = A^+ + (I - A^+A)U$, where $U \in \mathbb{C}^{n \times m}$ is arbitrary ([4, Theorem 2.5.17]). We will use the following simple fact ([4, Theorem 2.5.28 (iv)]): $A^+ = (A^*A)^+A^*$.

We shall mostly be concerned with core matrices. Recall that a square matrix $A$ is said to be core if $R(A)$ and $N(A)$ are complementary subspaces, which is equivalent to saying that $R(A) = R(A^2)$. Given a core matrix $A$, we let $Q_A$ represent the projector which projects a vector on $R(A)$ along $N(A)$. A $c$-inverse $A_c^-$ of a core matrix $A$ is defined to be a solution to the matrix equation $XA = Q_A$ ([4, Definition 6.4.1]). We let $\{ A_c^- \}$ denote the collection of all $A_c^-$. Among the $c$-inverses, those having $R(A^-_c) = R(A)$ are called $\chi$-inverses ([4, Definition 2.4.1]). According to Theorem 2.4.3 and Remark 2.4.14 of [4], the group inverse $A^\#$ is the uniquely determined $\chi$-inverse satisfying the following condition $N(A^\#) = N(A)$. It is evident that $A^\#$ is a reflexive generalized inverse of $A$ such that $AA^\# = A^\#A$ ([4, Theorem 2.4.6]).

Following [2], we define the core inverse $A^\oplus$ by $A^\oplus = A^\#AA^+$. In fact, $A^\oplus$ is the unique generalized inverse of $A$, which is both a least squares inverse and a $\chi$-inverse of $A$. In [2] there are presented some results on characterizations of $A^\oplus$. Finally, let us point out that the core inverse coincides with the hybrid inverse $A^{\rho*\chi}$ defined by Rao and Mitra [5, Section 4.10.2].

2. Core matrix partial order

We will be concerned here with the core relation defined by Baksalary and Trenkler [2].

**Definition 1.** For a pair of core matrices $A, B \in \mathbb{C}^{n \times n}$ we define the core relation $<^\oplus$ by saying that $A <^\oplus B$ if the following condition is satisfied:

\[
A^\oplus(B - A) = (B - A)A^\oplus = 0.
\]

The lemma below gives two other conditions that are equivalent to (1).

**Lemma 2.** Let $A$ and $B$ be core matrices of the same order. Then the following statements are equivalent:
1. \( A \prec B \),
2. \( A^+(B - A) = (B - A)A^# = 0 \),
3. \( A^*A = A^*B \) and \( BA = A^2 \).

**Proof.** We first recall the well-known fact ([3, Fact 2.10.12]) that \( \text{rank}(AB) = \text{rank}(A) \) if and only if \( R(AB) = R(A) \). This result implies, and is in fact equivalent to, the statement that \( \text{rank}(AB) = \text{rank}(B) \) if and only if \( N(AB) = N(B) \).

To establish the claim, observe that \( A\prec, A^+, A^\# \) and \( A \) have the same rank. Hence, \( R(A\prec) = R(A^+) = R(A) \) and \( N(A\prec) = N(A^+) = N(A^*) \), from which the required result follows.

Let us mention here another equivalent formulation of condition (1). As observed in [2, (3.21)], \( A \prec B \) if and only if \( A^+B = A^+A \) and \( BA = A^2 \).

Another concept referred to is the minus partial ordering (see, for example, [4, Chapter 3]). We say that \( A \prec - B \) if and only if \( (A-B)A^- = 0 \) and \( A^-(A-B) = 0 \) for some generalized inverse \( A^- \).

It is worth making the following Proposition, which includes Theorem 8 in [2].

**Proposition 3.** If \( A \prec B \) then \( A \prec - B \), \( R(A) \subset R(B) \), \( R(A^*) \subset R(B^*) \). The relation \( \prec \) is reflexive and antisymmetric.

The following Theorem describes a new property of the core relation \( \prec \).

**Theorem 4.** \( A \prec B \) if and only if \( \{B^{-}_{\ell}\} \subset \{A^{-}_{\ell}\} \) and \( \{B^{+}_{\ell}\} \subset \{A^{+}_{\ell}\} \).

**Proof.** For proof of necessity, assume that \( G \in \{B^{-}_{\ell}\} \). Since \( A \prec B \), we have \( A^*A = A^*B \) and \( R(A) \subset R(B) \). Therefore \( A^*AG = A^*BB^+ = A^* \). Premultiplying this relationship by \( A(A^*)^+ \) yields \( AG = AA^+ \), which justifies \( \{B^{-}_{\ell}\} \subset \{A^{-}_{\ell}\} \). Suppose next that \( G \in \{B^{+}_{\ell}\} \). Since \( BA = A^2 \), we get \( GA = GA^2A^\# = GBAA^\# = Q BA A^\# = AA^\# \). This proves that \( \{B^{+}_{\ell}\} \subset \{A^{+}_{\ell}\} \).

To show sufficiency, note that our assumption \( \{B^{-}_{\ell}\} \subset \{A^{-}_{\ell}\} \) forces \( A = B^\#A^\# \). Then, clearly, \( R(A) \subset R(B) \), and consequently, \( BA = BB^\#A^\# = A^2 \), as needed. Next, to establish \( A^*A = A^*B \), we consider the general expression \( B^{-}_{\ell} = B^+ + (I - B^+B)U \). If \( \{B^{-}_{\ell}\} \subset \{A^{-}_{\ell}\} \), then \( AB^{-}_{\ell} = AB^+ \),
and consequently, \( A(I - B^+B)U = 0 \) for every \( U \in \mathbb{C}^{n \times n} \), which implies that \( A = AB^+B \). Hence \( R(A^*) \subset R(B^*) \). Moreover, \( \{B^*_\ell\} \subset \{A^*_\ell\} \) guarantees that \( A^* = A^*AB^+ \). Therefore \( A^*B = A^*AB^+B = A^*A \), as required.

Theorem 4 guarantees that the core relation is transitive. On account of Proposition 3, we obtain that the relation \( \triangleleft \) defines a matrix partial ordering ([2, Theorem 6]).

In the following we shall link different types of partial orders together. The following terminology will be required ([4, Definitions 6.3.1, 6.5.2]).

For \( A, B \in \mathbb{C}^{m \times n} \), we define the left star relation \( * \) by saying that \( A^* < B \) if \( R(A) \subset R(B) \) and \( A^*A = A^*B \).

For core matrices \( A, B \in \mathbb{C}^{n \times n} \) we define the right sharp relation \( \# \) by setting \( A < \#B \) if \( R(A^*) \subset R(B^*) \) and \( A^2 = BA \).

The star relation is due to Baksalary and Mitra [1]. As is well known, the left star and the right sharp relation are partial orders ([1], [4, Corollary 6.3.10]).

Proposition 3 permits us to conclude with the following

**Proposition 5.** \( A \triangleleft B \) if and only if \( A^* < B \) and \( A < \#B \).

As a matter of fact, Proposition 5 states that the core relation is an intersection partial ordering ([4, Definition A.8.1]).

Some remarks are due. It was our intention here to present a fairly simple and self-contained proof of Theorem 4. However, once Proposition 5 is established, Theorem 4 may be achieved by appealing to characterizations of one-sided orders as given by Theorems 6.4.8 and 6.5.17 in [4].

**References**


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