PRE-STRONGLY SOLID VARIETIES
OF COMMUTATIVE SEMIGROUPS

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AND

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Abstract

Generalized hypersubstitutions are mappings from the set of all fundamental operations into the set of all terms of the same language do not necessarily preserve the arities. Strong hyperidentities are identities which are closed under the generalized hypersubstitutions and a strongly solid variety is a variety which every its identity is a strong hyperidentity. In this paper we give an example of pre-strongly solid varieties of commutative semigroups and determine the least and the greatest pre-strongly solid variety of commutative semigroups.

Keywords and phrases: generalized hypersubstitution, pre-strongly solid variety, commutative semigroup.

2000 Mathematics Subject Classification: 20M07, 08B15, 08B25.

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1. Introduction

Hyperidentities were invented by Aczel, Belousov and Taylor. The notion of \textit{hyperidentities} and \textit{solid varieties of a given type} were invented by E. Graczyńska and D. Schweigert in [3]. An identity \( t \approx t' \) of terms of any type \( \tau \) is called a \textit{hyperidentity} for an algebra \( A = (A; (f_i^A)_{i \in I}) \) if \( t \approx t' \) holds identically for every choice of \( n \)-ary term operation to represent \( n \)-ary operation symbols occurring in \( t \) and \( t' \). A variety which every its identity is a hyperidentity is called \textit{solid variety}. Hyperidentities can be characterized more precisely using the concept of a \textit{hypersubstitution} which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert. A hypersubstitution of type \( \tau \) is a mapping \( \sigma : \{ f_i \mid i \in I \} \rightarrow W_\tau(X) \) which assigns to every \( n_i \)-ary operation symbol \( f_i \) an \( n_i \)-ary term. The set of all hypersubstitutions of type \( \tau \) is denoted by \( \text{Hyp}(\tau) \).

For every \( \sigma \in \text{Hyp}(\tau) \) induces a mapping \( \hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X) \) by the following steps:

(i) \( \hat{\sigma}[x] := x \), for any variable \( x \in X \), and

(ii) \( \hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]) \), where \( \hat{\sigma}[t_j], 1 \leq j \leq n_i \) are already defined.

A binary operation \( \circ_h \) on \( \text{Hyp}(\tau) \) is defined by \( \sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \hat{\sigma}_2 \) for every \( \sigma_1, \sigma_2 \in \text{Hyp}(\tau) \) where \( \circ \) is the natural composition of mappings. Let \( \sigma_{id} \) be the hypersubstitution where \( \sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i}) \). It turns out that \( (\text{Hyp}(\tau); \circ_h, \sigma_{id}) \) is a monoid with \( \sigma_{id} \) is an identity element.

S. Leeratanavalee and K. Denecke generalized the concepts of hypersubstitutions, hyperidentities and solid varieties to generalized hypersubstitutions, strong hyperidentities and strongly solid varieties [4]. A generalized hypersubstitution of type \( \tau \) is a mapping \( \sigma : \{ f_i \mid i \in I \} \rightarrow W_\tau(X) \) from the set of all \( n_i \)-ary operation symbols into the set of all terms built up by elements of the alphabet \( X := \{ x_1, x_2, \ldots \} \) and operation symbols from \( \{ f_i \mid i \in I \} \) which does not necessarily preserve the arity.

We denoted the set of all generalized hypersubstitutions of type \( \tau \) by \( \text{Hyp}_G(\tau) \). To define a binary operation on \( \text{Hyp}_G(\tau) \), we defined firstly the concept of generalized superposition of terms \( S^m : W_\tau(X)^{m+1} \rightarrow W_\tau(X) \) by the following steps:

for any term \( t \in W_\tau(X) \),
(i) if $t = x_j, 1 \leq j \leq m$, then

$$S^m(x_j, t_1, \ldots, t_m) := t_j,$$

(ii) if $t = x_j, m < j \in \mathbb{N}$, then

$$S^m(x_j, t_1, \ldots, t_m) := x_j,$$

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then

$$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$$.

Then the generalized hypersubstitution $\sigma$ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^m_i(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any $n_i$-ary operation symbol $f_i$ where $\hat{\sigma}[t_j], 1 \leq j \leq n_i$ are already defined.

We defined a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \hat{\sigma}_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let $\sigma_{id}$ be the hypersubstitution mapping which maps each $n_i$-ary operation symbol $f_i$ to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $(Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the monoid $(Hyp(\tau); \circ_G, \sigma_{id})$ of all arity preserving hypersubstitutions of type $\tau$ forms a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$.

If $M$ is a submonoid of $Hyp_G(\tau)$ then an identity $t \approx t'$ is called an $M$-strong hyperidentity if $\hat{\sigma}[t] \approx \hat{\sigma}[t']$ are identities for every $\sigma \in M$. A variety $V$ is called $M$-strongly solid if every identity in it is an $M$-strong hyperidentity. In case of $M = Hyp_G(\tau)$ we will call a strong hyperidentity and strongly solid respectively.

2. $V$-proper generalized hypersubstitutions and normal forms

Definition 2.1 ([5]). Let $V$ be a variety of type $\tau$. A generalized hyper-substitution $\sigma$ of type $\tau$ is called a $V$-proper generalized hypersubstitution if for every identity $s \approx t$ of $V$, the identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ also holds in $V$. We use $P_G(V)$ for the set of all $V$-proper generalized hypersubstitutions of type $\tau$.

Proposition 2.2 ([5]). For any variety $V$ of type $\tau$, $(P_G(V); \circ_G, \sigma_{id})$ is a submonoid of $(\text{Hyp}_G(\tau); \circ_G, \sigma_{id})$.

Definition 2.3 ([5]). Let $V$ be a variety of type $\tau$. Two generalized hypersubstitutions $\sigma_1$ and $\sigma_2$ of type $\tau$ are called a $V$-generalized equivalent if $\sigma_1(f_i) \approx \sigma_2(f_i)$ are identities in $V$ for all $i \in I$. In this case we write $\sigma_1 \sim_{VG} \sigma_2$.

Theorem 2.4 ([5]). Let $V$ be a variety of algebras of type $\tau$, and let $\sigma_1, \sigma_2 \in \text{Hyp}_G(\tau)$. Then the following statements are equivalent:

(i) $\sigma_1 \sim_{VG} \sigma_2$.

(ii) For all $t \in W_\tau(X)$, the equations $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ are identities in $V$.

(iii) For all $A \in V$, $\sigma_1[A] = \sigma_2[A]$ where $\sigma_k[A] = (A; (f_i)^A_{i \in I}); k = 1, 2$.

Proposition 2.5 ([5]). Let $V$ be a variety of algebras of type $\tau$. Then the following statements hold:

(i) For all $\sigma_1, \sigma_2 \in \text{Hyp}_G(\tau)$, if $\sigma_1 \sim_{VG} \sigma_2$ then $\sigma_1$ is a $V$-proper generalized hypersubstitution iff $\sigma_2$ is a $V$-proper generalized hypersubstitution.

(ii) For all $s, t \in W_\tau(X)$ and for all $\sigma_1, \sigma_2 \in \text{Hyp}_G(\tau)$, if $\sigma_1 \sim_{VG} \sigma_2$ then $\hat{\sigma}_1[s] \approx \hat{\sigma}_2[t]$ is an identity in $V$ iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ is an identity in $V$.

The relation $\sim_{VG}$ is an equivalence relation on $\text{Hyp}_G(\tau)$, but it is not necessary a congruence relation. We factorize $\text{Hyp}_G(\tau)$ by $\sim_{VG}$ and consider the submonoid $P_G(V)$ of $\text{Hyp}_G(\tau)$ is the union of equivalence classes of the relation $\sim_{VG}$. This is also true for a submonoid $M$ of $\text{Hyp}_G(\tau)$ and the relation $\sim_{VG|M}$.

Lemma 2.6 ([5]). Let $M$ be a submonoid of $\text{Hyp}_G(\tau)$ and let $V$ be a variety of type $\tau$. Then the monoid $P_G \cap M$ is the union of all equivalence classes of the restricted relation $\sim_{VG|M}$.
Definition 2.7 ([5]). Let $M$ be a monoid of generalized hypersubstitutions of type $\tau$, and let $V$ be a variety of type $\tau$. Let $\phi$ be a choice function which chose from $M$ one generalized hypersubstitution from each equivalence class of the relation $\sim_{V|G_{M}}$, and let $N_{\phi}^{M}(V)$ be the set of generalized hypersubstitutions which are chosen. Thus $N_{\phi}^{M}(V)$ is a set of distinguished generalized hypersubstitutions from $M$, which we might call $V$-normal form generalized hypersubstitutions. We will say that the variety $V$ is $N_{\phi}^{M}(V)$-strongly solid if for every identity $s \approx t \in IdV$ and for every generalized hypersubstitution $\sigma \in N_{\phi}^{M}(V)$, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$.

Theorem 2.8 ([5]). Let $M$ be a monoid of generalized hypersubstitutions of type $\tau$ and let $V$ be a variety of type $\tau$. For any choice function $\phi$, $V$ is $M$-strongly solid if and only if $V$ is $N_{\phi}^{M}(V)$-strongly solid.

3. Pre-strongly solid varieties of semigroups

The concept of pre-solid varieties was introduced by K. Denecke and S.L. Wismath [2]. In 2007, S. Leeratanavalee and S. Phatcharagern generalized the concept of pre-solid varieties to pre-strongly solid varieties [5]. Firstly, we recall the definitions of a pre-generalized hypersubstitution and a pre-strong hyperidentity. Let us fix a type $\tau = (2)$. So we have only one binary operation symbol, say $f$. From now on, the generalized hypersubstitution $\sigma$ which maps $f$ to the term $t$ is denoted by $\sigma_{t}$.

Definition 3.1. A generalized hypersubstitution $\sigma \in HypG(2)$ is called a pre-generalized hypersubstitution if $\sigma \in HypG(2) \setminus \{\sigma_{x_{1}}, \sigma_{x_{2}}\}$ where $\sigma_{x_{1}}$ and $\sigma_{x_{2}}$ denoted the generalized hypersubstitutions which map $f$ to $x_{1}$ and to $x_{2}$, respectively. We denote the set of all pre-generalized hypersubstitutions of type $\tau = (2)$ by $PreG(2)$.

The reason to delete the generalized hypersubstitutions $\sigma_{x_{1}}$ and $\sigma_{x_{2}}$ from $HypG(2)$ is if we apply the generalized hypersubstitution $\sigma_{x_{1}}$ or $\sigma_{x_{2}}$ on the both sides of the commutative law $x_{1}x_{2} \approx x_{2}x_{1}$ we obtain the equation $x_{1} \approx x_{2}$ which satisfied only in a one-element semigroup.

Definition 3.2. An identity $t \approx t'$ is called a pre-strong hyperidentity in a variety $V$ if $\hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV$ for all $\sigma \in PreG(2)$.

A variety $V$ is called a pre-strongly solid variety if every identity in $V$ is a pre-strong hyperidentity of $V$. 
For a class $K$ of algebras of type $\tau$ and for a set $\sum$ of identities of this type we fix the following notations:

- $IdK$ - the set of all identities of $K$,
- $HIdK$ - the set of all hyperidentities of $K$,
- $H_{PreG}IdK$ - the set of all pre-strong hyperidentities of $K$,
- $Mod\sum = \{ A \in Alg(\tau) | A \text{ satisfies } \sum \}$ - the variety defined by $\sum$,
- $HMod\sum = \{ A \in Alg(\tau) | A \text{ hypersatisfies } \sum \}$ - the hyperequational class defined by $\sum$,
- $H_{PreG}Mod\sum = \{ A \in Alg(\tau) | A \text{ pre-strong hypersatisfies } \sum \}$ - the pre-strong hyperequational class defined by $\sum$.

**Proposition 3.3** ([5]). $PreG(2)$ is a submonoid of $HypG(2)$.

**Remark 3.4** ([5]). Every strongly solid variety of semigroups is a pre-strongly solid variety.

**Remark 3.5** ([5]). Every pre-strongly solid variety of semigroups is a pre-solid variety of semigroups.

**Lemma 3.6** ([5]). The variety $Z := Mod\{ x_1x_2 \approx x_3x_4 \}$ is the least non-trivial pre-strongly solid variety of semigroups.

**Theorem 3.7** ([5]). The greatest non-trivial pre-strongly solid variety of semigroups which is not strongly solid is $Z := Mod\{ x_1x_2 \approx x_3x_4 \}$.

**Theorem 3.8** ([5]). The variety $V_{big} := Mod\{ (x_1x_2)x_3 \approx x_1(x_2x_3), x_1^2x_2 \approx x_1x_2, x_1x_2x_3x_4 \approx x_1x_3x_2x_4 \}$ is the greatest pre-strongly solid variety of semigroups.

4. **Pre-strongly solid varieties of commutative semigroups**

Firstly, we recall the definition of a generalized hypersubstitution of type $\tau$ is a mapping $\sigma : \{ f_i | i \in I \} \rightarrow W_\tau(X)$ from the set of all $n_i$-ary operation symbols into the set of all terms built up by elements of the alphabet.
$X := \{x_1, x_2, \ldots\}$ and operation symbols from $\{f_i | i \in I\}$ which does not necessarily preserve the arity. We denote the set of all generalized hyper-substitutions of type $\tau$ by $Hyp_G(\tau)$. A generalized superposition of terms $S^m : W_\tau(X)^{m+1} \rightarrow W_\tau(X)$ is defined by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq m$, then

$$S^m(x_j, t_1, \ldots, t_m) := t_j,$$

(ii) if $t = x_j, m < j \in \mathbb{N}$, then

$$S^m(x_j, t_1, \ldots, t_m) := x_j,$$

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then

$$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m)).$$

For every $\sigma \in Hyp_G(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any $n_i$-ary operation symbol $f_i$ where $\hat{\sigma}[t_j], 1 \leq j \leq n_i$ are already defined.

In this section, we give an example of pre-strongly solid varieties of commutative semigroups and then determine the least and the greatest pre-strongly solid variety of commutative semigroups.

**Theorem 4.1.** The variety $V_1 := Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1, x_1x_2 \approx x_1x_2, x_1^2 \approx x_2^2\}$ is a pre-strongly solid variety of commutative semigroups.

**Proof.** To show that the variety $V_1$ is a pre-strongly solid variety of commutative semigroups, we have to show that every identity satisfied in $V_1$ is a pre-strong hyperidentity of $V_1$. By using Theorem 2.8, we can restrict our checking to the following pre-generalized hypersubstitutions $\sigma_t$ where $t \in \{x_ix_j | i, j \in \mathbb{N}\} \cup \{x_ix_jx_k | i \neq j \neq k\} \cup \{x_{i_1}x_{i_2} \ldots x_{i_k} | k, i_1, \ldots, i_k \in \mathbb{N}, k > 3, \text{ and all of } i_1, \ldots, i_k \text{ are distinct}\}$. 
If we apply $\sigma_{x_i x_j}; i, j \in \mathbb{N}$ on the both sides of the associative law we have the following table.

<table>
<thead>
<tr>
<th>$i, j \in \mathbb{N}$</th>
<th>$\hat{\sigma}_{x_i x_j}[(x_1 x_2) x_3] = S^2(x_i x_j, S^2(x_1 x_2, x_3))$</th>
<th>$\hat{\sigma}_{x_i x_j}[x_1 (x_2 x_3)] = S^2(x_i x_j, x_1, S^2(x_1 x_2, x_3))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = j = 1$</td>
<td>$x_1 x_1 x_1 x_1$</td>
<td>$x_1 x_1$</td>
</tr>
<tr>
<td>$i = 1, j = 2$</td>
<td>$x_1 x_2 x_3$</td>
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<tr>
<td>$i = 1, j &gt; 2$</td>
<td>$x_1 x_j x_j$</td>
<td>$x_1 x_j$</td>
</tr>
<tr>
<td>$i = j = 2$</td>
<td>$x_3 x_3$</td>
<td>$x_3 x_3 x_3 x_3$</td>
</tr>
<tr>
<td>$i = 2, j &gt; 2$</td>
<td>$x_3 x_j x_j$</td>
<td>$x_3 x_j x_j$</td>
</tr>
<tr>
<td>$i, j &gt; 2$</td>
<td>$x_i x_j$</td>
<td>$x_i x_j$</td>
</tr>
</tbody>
</table>

Using the associative law, the commutative law and identities $x_1^2 x_2 \approx x_1 x_2^2 \approx x_1 x_2, x_1^2 \approx x_2^2$ we have both sides are equal.

If we apply $\sigma_{x_i x_j}; i, j \in \mathbb{N}$ on the both sides of the commutative law we have the following table.

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If we apply $\sigma_{i,j}; i, j \in \mathbb{N}$ on the both sides of the identity $x_1^2 \approx x_2^2$ we have the following table.

<table>
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<tr>
<th>$i, j \in \mathbb{N}$</th>
<th>$\hat{\sigma}_{i,j} [x_1 x_1] = S^2(x_i x_j, x_1, x_1)$</th>
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If we apply $\sigma_{x_1 x_2 x_k}; i \neq j \neq k \in \mathbb{N}$ on the both sides of the associative law we have the following table.

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<tr>
<th>$i, j, k \in \mathbb{N}$</th>
<th>$\hat{\sigma}_{x_1 x_2 x_k}[x_1 x_2] = S^2(x_1 x_2 x_k, S^2(x_1 x_2 x_k, x_1, x_2), x_3)$</th>
<th>$\hat{\sigma}_{x_1 x_2 x_k}[x_2 x_1] = S^2(x_1 x_2 x_k, x_1, S^2(x_1 x_2 x_k, x_2, x_3))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1, j = 2, k \geq 2$</td>
<td>$x_1 x_2 x_k x_3 x_k$</td>
<td>$x_1 x_2 x_k x_3 x_k$</td>
</tr>
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If we apply $\sigma_{x_1 x_2 x_k}; i \neq j \neq k \in \mathbb{N}$ on the both sides of the commutative law we have the following table.

<table>
<thead>
<tr>
<th>$i, j, k \in \mathbb{N}$</th>
<th>$\hat{\sigma}_{x_1 x_2 x_k}[x_1 x_2] = S^2(x_1 x_2 x_k, x_1, x_2)$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$i = 1, j = 2, k \geq 2$</td>
<td>$x_1 x_2 x_k$</td>
<td>$x_2 x_1 x_k$</td>
</tr>
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<td>$x_2 x_1 x_k$</td>
</tr>
<tr>
<td>$i = 2, j, k \geq 2$</td>
<td>$x_2 x_1 x_k$</td>
<td>$x_2 x_1 x_k$</td>
</tr>
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If we apply $\sigma_{x_i x_j x_k} ; i \neq j \neq k \in \mathbb{N}$ on the both sides of the identity $x_1^2 \approx x_2^2$ we have the following table.

<table>
<thead>
<tr>
<th>$i, j, k \in \mathbb{N}$</th>
<th>$\hat{\sigma}_{x_i x_j x_k} [x_1 x_1] = S^2(x_1 x_j x_k, x_1, x_1)$</th>
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If we apply $\sigma_{x_i x_j x_k} ; i \neq j \neq k \in \mathbb{N}$ on the both sides of the identity $x_1^2 x_2 \approx x_1 x_2^2 \approx x_1 x_2$ we have the following table.

<table>
<thead>
<tr>
<th>$i, j, k \in \mathbb{N}$</th>
<th>$\hat{\sigma}_{x_i x_j x_k} [(x_1 x_1)x_2] = S^2((x_1 x_j x_k), x_1, x_2)$</th>
<th>$\hat{\sigma}_{x_i x_j x_k} [(x_2 x_2)x_1] = S^2((x_1 x_j x_k), x_2, x_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1, j = 2, k &gt; 2$</td>
<td>$x_1 x_1 x_k x_2 x_k$</td>
<td>$x_1 x_2 x_2 x_k x_k$</td>
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<td>$i = 2, j, k &gt; 2$</td>
<td>$x_2 x_j x_k$</td>
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Using the associative law, the commutative law and identities $x_1^2 x_2 \approx x_1 x_2^2 \approx x_1 x_2, x_1^2 \approx x_2^2$ we have both sides are equal.
If we apply $\sigma_t$ where $t = x_{i_1}x_{i_2}...x_{i_k}$ and $k, i_1, ..., i_k \in \mathbb{N}, k > 3$ on the both sides of the associative law we have $\hat{\sigma}_t((x_1x_2)x_3) = S^2(t, S^2(t, x_1, x_2), x_3)$ and $\hat{\sigma}_t(x_1(x_2x_3)) = S^2(t, x_1, S^2(t, x_2, x_3))$.

(i) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and $i_m > 2$ for all $m \neq n$, then

$$\hat{\sigma}_t((x_1x_2)x_3) = x_{i_1}...x_{i_{n-1}}x_{i_1}...x_{i_{n-1}}x_{i_{n+1}}...x_{i_k}x_{i_{n+1}}...x_{i_k};$$

$$\hat{\sigma}_t(x_1(x_2x_3)) = x_{i_1}...x_{i_{n-1}}x_{i_1}x_{i_{n+1}}...x_{i_k}x_{i_{n+1}}...x_{i_k}.$$  

(ii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 2$ and $i_m > 2$ for all $m \neq n$, then

$$\hat{\sigma}_t((x_1x_2)x_3) = x_{i_1}...x_{i_{n-1}}x_3x_{i_{n+1}}...x_{i_k};$$

$$\hat{\sigma}_t(x_1(x_2x_3)) = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}x_{i_{n+1}}...x_{i_k}.$$  

(iii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and there exists a unique $l \in \{1, ..., k\}$ such that $i_l = 2$, $i_m > 2$ for all $m \neq n \neq l$ and $n < l$, then

$$\hat{\sigma}_t((x_1x_2)x_3) = x_{i_1}...x_{i_{n-1}}x_{i_1}x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_{l+1}}...x_{i_k}x_{i_{l+1}}...x_{i_k};$$

$$\hat{\sigma}_t(x_1(x_2x_3)) = x_{i_1}...x_{i_{n-1}}x_{i_1}x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_{l+1}}...x_{i_k}x_{i_{l+1}}...x_{i_k}.$$  

(iv) If $i_m > 2$ for all $m \in \{1, 2, ..., k\}$, then

$$\hat{\sigma}_t((x_1x_2)x_3) = x_{i_1}...x_{i_k};$$ 

$$\hat{\sigma}_t(x_1(x_2x_3)) = x_{i_1}...x_{i_k}.$$ 

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2$ we have both sides are equal.
If we apply $\sigma_t$ where $t = x_1x_2...x_i$ and $k, i_1, ..., i_k \in \mathbb{N}, k > 3$ on the both sides of the commutative law we have $\hat{\sigma}_t[x_1x_2] = S^2(t, x_1, x_2)$ and $\hat{\sigma}_t[x_2x_1] = S^2(t, x_2, x_1)$.

(i) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and $i_m > 2$ for all $m \neq n$, then

\[\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k},\]
\[\hat{\sigma}_t[x_2x_1] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_k}.\]

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(iii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and there exists a unique $l \in \{1, ..., k\}$ such that $i_l = 2$, $i_m > 2$ for all $m \neq n \neq l$ and $n < l$, then

\[\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_l+1}...x_{i_k},\]
\[\hat{\sigma}_t[x_2x_1] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_{l-1}}x_1x_{i_l+1}...x_{i_k}.\]

(iv) If $i_m > 2$ for all $m \in \{1, 2, ..., k\}$, then

\[\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_k},\]
\[\hat{\sigma}_t[x_2x_1] = x_{i_1}...x_{i_k}.\]

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2$, $x_1^2 \approx x_2^2$ we have both sides are equal.
If we apply $\sigma_t$ where $t = x_{i_1}x_{i_2}...x_{i_k}$ and $k, i_1, ..., i_k \in \mathbb{N}, k > 3$ on the both sides of the identity $x_1^2 \approx x_2^2$ we have $\hat{\sigma}_t[x_1x_1] = S^2(t, x_1, x_1)$ and $\hat{\sigma}_t[x_2x_2] = S^2(t, x_2, x_2)$.

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$\hat{\sigma}_t[x_2x_2] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_k}$.

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If we apply \( \sigma_t \) where \( t = x_1x_2...x_{i_k} \) and \( k, i_1, ..., i_k \in \mathbb{N}, k > 3 \) on the both sides of the identity \( x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2 \) we have \( \hat{\sigma}_t[(x_1x_1)x_2] = S^2(t, S^2(t, x_1, x_1), x_2) \) and \( \hat{\sigma}_t[x_2(x_2x_2)] = S^2(t, x_1, S^2(t, x_1, x_2)) \) and \( \hat{\sigma}_t[x_1x_2] = S^2(t, x_1, x_2) \).

(i) If there exists a unique \( n \in \{1, ..., k\} \) such that \( i_n = 1 \) and \( i_m > 2 \) for all \( m \neq n \), then

\[
\hat{\sigma}_t[(x_1x_1)x_2] = x_{i_1}...x_{i_{n-1}}x_{i_1}x_1x_{i_{n+1}}...x_kx_{i_{n+1}}...x_{i_k}.
\]

\[
\hat{\sigma}_t[x_2(x_2x_2)] = x_{i_1}...x_{i_{n-1}}x_{i_1}x_1x_{i_{n+1}}...x_kx_{i_{n+1}}...x_{i_k}.
\]

\[
\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}.
\]

(ii) If there exists a unique \( n \in \{1, ..., k\} \) such that \( i_n = 2 \) and \( i_m > 2 \) for all \( m \neq n \), then

\[
\hat{\sigma}_t[(x_1x_1)x_2] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_kx_{i_{n+1}}...x_{i_k}.
\]

\[
\hat{\sigma}_t[x_2(x_2x_2)] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_kx_{i_{n+1}}...x_{i_k}.
\]

\[
\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_k}.
\]

(iii) If there exists a unique \( n \in \{1, ..., k\} \) such that \( i_n = 1 \) and there exists a unique \( l \in \{1, ..., k\} \) such that \( i_l = 2 \), \( i_m > 2 \) for all \( m \neq n \neq l \) and \( n < l \), then

\[
\hat{\sigma}_t[(x_1x_1)x_2] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_{l-1}}x_1x_{i_{l+1}}...x_kx_{i_{l+1}}...x_{i_{n-1}}x_{i_{n+1}}...x_{i_k}.
\]

\[
\hat{\sigma}_t[x_2(x_2x_2)] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_{l+1}}...x_kx_{i_{l+1}}...x_{i_{n-1}}x_{i_{n+1}}...x_{i_k}.
\]

\[
\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_{l+1}}...x_{i_k}.
\]
Theorem 4.3. The variety \( V_2 := \text{Mod}\{x_1x_2 \approx x_3x_4\} \) is the greatest pre-strongly solid variety of commutative semigroups.

Proof. The greatest pre-strongly solid variety of commutative semigroups is the class of all commutative semigroups for which the associative law and the commutative law are satisfied as pre-strong hyperidentities, i.e. the class \( H_{Pr\text{reg}}\text{Mod}\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1, x_1x_2x_3 \approx x_1x_3\} \). Applying \( \sigma_{x_1x_2}, \sigma_{x_1x_3}, \sigma_{x_2x_1} \) \((i > 2) \in Pr_{\text{reg}}\) on the associative law, \( \sigma_{x_1x_2} \) gives \( (x_1x_2)x_3 \approx x_1(x_2x_3), \) \( \sigma_{x_1x_3} \) gives \( x_1x_2x_3 \approx x_1x_3, \) \( \sigma_{x_2x_1} \) gives \( x_2x_1x_3 \approx x_2x_3. \) If we substitute for \( x_i \) a new variable \( x_2, \) then we have the identities \( x_1x_2^2 \approx x_1x_2, x_2^2x_1 \approx x_2x_1. \) That means \( x_1x_2^2 \approx x_1x_2 \approx x_1x_2 \in \text{Id}(H_{Pr\text{reg}}\text{Mod}\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\}) \). Applying \( \sigma_{x_1x_2}, \sigma_{x_1x_3}, \sigma_{x_2x_1} \) \((i > 2) \) on the commutative law, \( \sigma_{x_1x_2} \) gives \( x_1x_2 \approx x_2x_1, \sigma_{x_1x_3} \) gives \( x_1x_3 \approx x_1x_2. \) Then \( x_1x_2x_2 \approx x_1x_2, x_1x_2 \approx x_2x_2, \) so \( x_1x_2 \approx x_2x_2. \) If we substitute \( x_i \) by \( x_1, x_2 \) by \( x_2 \) by \( x_3. \) Then we have \( x_1x_2x_3 \approx x_1x_3. \) Thus \( H_{Pr\text{reg}}\text{Mod}\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\} \) satisfies all identities of \( V_2, \) i.e \( H_{Pr\text{reg}}\text{Mod}\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\} \subseteq V_2. \) To prove the converse inclusion we have to check the associative law, the commutative law and the rectangular law, i.e. \( x_1x_2x_3 \approx x_1x_3 \) using all pre-generalized hypersubstitutions. We can restrict our checking to the following pre-generalized hypersubstitutions \( \sigma_{x_i,x_j}(i, j \in \mathbb{N}). \)
If we apply $\sigma_{x_i x_j}; i, j \in \mathbb{N}$ on the both sides of the associative law we have the following table.

<table>
<thead>
<tr>
<th>$i, j \in \mathbb{N}$</th>
<th>$\hat{\sigma}_{x_i x_j}[(x_1 x_2) x_3] = S^2(x_i x_j, S^2(x_1 x_i, x_2), x_3)$</th>
<th>$\hat{\sigma}_{x_i x_j}[x_1 (x_2 x_3)] = S^2(x_i x_j, x_1, S^2(x_i x_j, x_2, x_3))$</th>
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<tr>
<td>$i = j = 1$</td>
<td>$x_1 x_1 x_1$</td>
<td>$x_1 x_1$</td>
</tr>
<tr>
<td>$i = 1, j = 2$</td>
<td>$x_1 x_2 x_3$</td>
<td>$x_1 x_2 x_3$</td>
</tr>
<tr>
<td>$i = j = 2$</td>
<td>$x_3 x_3$</td>
<td>$x_3 x_3 x_3 x_3$</td>
</tr>
<tr>
<td>$i = 1, j &gt; 2$</td>
<td>$x_1 x_j x_j$</td>
<td>$x_1 x_j$</td>
</tr>
<tr>
<td>$i = 2, j &gt; 2$</td>
<td>$x_3 x_j$</td>
<td>$x_3 x_j x_j$</td>
</tr>
<tr>
<td>$i, j &gt; 2$</td>
<td>$x_i x_j$</td>
<td>$x_i x_j$</td>
</tr>
</tbody>
</table>

Using the associative law, the commutative law and the identity $x_1 x_2 x_3 \approx x_1 x_3$ we have both sides are equal.

If we apply $\sigma_{x_i x_j}; i, j \in \mathbb{N}$ on the both sides of the commutative law we have the following table.

<table>
<thead>
<tr>
<th>$i, j \in \mathbb{N}$</th>
<th>$\hat{\sigma}_{x_i x_j}[x_1 x_2] = S^2(x_i x_j, x_1, x_2)$</th>
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<td>$i, j &gt; 2$</td>
<td>$x_i x_j$</td>
<td>$x_i x_j$</td>
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</table>

Using the associative law, the commutative law and the identity $x_1 x_2 x_3 \approx x_1 x_3$ we have both sides are equal.
If we apply $\sigma_{x, x_j} \colon i, j \in \mathbb{N}$ on the both sides of the identity $x_1 x_2 x_3 \approx x_1 x_3$ we have the following table.

<table>
<thead>
<tr>
<th>$i, j \in \mathbb{N}$</th>
<th>$\hat{\sigma}_{x, x_j}[(x_1 x_2)_3] = S^2(x, x_j, S^2(x, x_j, x_1, x_2), x_3)$</th>
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</table>

Using the associative law, the commutative law and the identity $x_1 x_2 x_3 \approx x_1 x_3$ we have both sides are equal.\[\blacksquare\]
Acknowledgements

This work was granted by one of the Higher Education Commission. Sarawut Phuapong and Sorasak Leeratanavalee were supported by CHE Ph.D Scholarship, the Graduate School and the Faculty of Science, Chiang Mai University, Thailand.

References


Received 25 December 2009
Revised 25 March 2010