CHARACTERIZATION OF TREES WITH EQUAL 2-DOMINATION NUMBER AND DOMINATION NUMBER PLUS TWO

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Abstract

Let $G = (V(G), E(G))$ be a simple graph, and let $k$ be a positive integer. A subset $D$ of $V(G)$ is a $k$-dominating set if every vertex of $V(G) - D$ is dominated at least $k$ times by $D$. The $k$-domination number $\gamma_k(G)$ is the minimum cardinality of a $k$-dominating set of $G$. In [5] Volkmann showed that for every nontrivial tree $T$, $\gamma_2(T) \geq \gamma_1(T) + 1$ and characterized extremal trees attaining this bound. In this paper we characterize all trees $T$ with $\gamma_2(T) = \gamma_1(T) + 2$.

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1. Introduction

In a graph \( G = (V(G), E(G)) = (V, E) \) of order \( n(G) \), or simply \( n \) when the graph \( G \) is clear from the context, the neighborhood \( N_G(v) = N(v) \) of a vertex \( v \in V(G) \) consists of the vertices adjacent with \( v \), and \( N_G[v] = N[v] = N(v) \cup \{v\} \) is the closed neighborhood. If \( S \) is a subset of vertices, then the subgraph induced by \( S \) in \( G \) is denoted \( G[S] \). The degree of a vertex \( v \), denoted by \( \deg_G(v) \), is the size of its open neighborhood. A vertex of degree one is called a leaf, and its neighbor is called a support vertex. We also denote the set of leaves of a graph \( G \) by \( L(G) \) and the set of support vertices by \( S(G) \). A tree \( T \) is a double star if it contains exactly two vertices that are not leaves. A double star with respectively \( p \) and \( q \) leaves attached at each support vertex is denoted by \( S_{p,q} \). The subdivision graph of a graph \( G \) is that graph obtained from \( G \) by replacing each edge \( uv \) of \( G \) by a vertex \( w \) and edges \( uw \) and \( vw \). If a tree \( T \) is a subdivision graph of a nontrivial tree \( T' \), then we say that \( T \) is a subdivided tree, and the \( n(T') - 1 \) new vertices resulting from the subdivision of the edges of \( T' \) are called subdivision vertices. Note that a subdivided tree has order at least three and at least one subdivision vertex. The corona graph \( G \circ K_1 \) of a graph \( G \) is the graph constructed from a copy of \( G \), where for each vertex \( v \in V(G) \), a new vertex \( v' \) and a pendant edge \( vv' \) are added. Let \( P_n \) denote the path graph of order \( n \).

Let \( k \) be a positive integer. A subset \( D \subseteq V(G) \) is a \( k \)-dominating set of the graph \( G \), if \( |N_G(v) \cap D| \geq k \) for every \( v \in V(G) - D \). The \( k \)-domination number \( \gamma_k(G) \) is the minimum cardinality among the \( k \)-dominating sets of \( G \). Note that the 1-domination number \( \gamma_1(G) \) is the usual domination number \( \gamma(G) \). A set \( S \subseteq V(G) \) is independent if no edge of \( G \) has its two endvertices in \( S \).

We make a couple of straightforward observations.

**Observation 1.** For every graph \( G \) and positive integer \( k \), every vertex with degree at most \( k - 1 \) belongs to every \( \gamma_k(G) \)-set.

**Observation 2.** For any tree \( T \) of order at least three, there exists a \( \gamma(T) \)-set that contains no leaves of \( T \).

The following results will be useful for the next.

**Theorem 3** (Fink and Jacobson [2] 1985). If \( T \) is a tree of order \( n \), then \( \gamma_2(T) \geq (n + 1)/2 \), with equality if and only if \( T = P_1 \) or \( T \) is the subdivided graph of another tree.
Theorem 4 (Volkmann [5] 2007). For every nontrivial tree \( T \), \( \gamma_2(T) \geq \gamma(T) + 1 \) with equality if and only if \( T \) is a subdivided star, the corona of a star, or a subdivided double star.

Let \( \mathcal{T} \) be the family of extremal trees achieving equality in Theorem 4, that is, \( \mathcal{T} \) is the family of nontrivial trees \( T \), where \( T \) is a subdivided star, the corona of a star, or a subdivided double star. For a subdivided tree in \( \mathcal{T} \), we let \( B(T) \) denote the set of subdivided vertices. Note that the corona of a star can also be described as a subdivided star with an added leaf adjacent to its center vertex. Thus, if \( T'' \) is the subdivision graph of a star \( T' \), then for the corona \( T \) of a star \( T' \), we let \( B(T) = B(T'') \). Note that the paths \( P_2 \) and \( P_4 \) are coronas of stars, and for the path \( P_2 \), \( B(T) = \emptyset \), and for the path \( P_4 \), \( B(T) \) consists of exactly one support vertex. For any tree in \( \mathcal{T} \), we let \( A(T) = V(T) - B(T) \). (Note that if \( T \) is a subdivision of a tree \( T' \), then \( A(T) = V(T') \) and if \( T \) is a corona, that is, a subdivision of a star \( T' \) with a leaf neighbor \( u \) added to its center, then \( A(T) = V(T') \cup \{u\} \)).

Thus, by Theorem 4, if \( T \) is a tree and \( T \) is not in \( \mathcal{T} \), then \( \gamma_2(T) \geq \gamma(T) + 2 \). Our aim in this paper is to characterize all trees \( T \) with \( \gamma_2(T) = \gamma(T) + 2 \).

We close this section by the following observation.

Observation 5. If \( T \in \mathcal{T} \), then \( A(T) \) is a \( \gamma_2(T) \)-set. Moreover, if \( T \in \mathcal{T} \) and \( T \neq P_4 \), then \( A(T) \) is the unique \( \gamma_2(T) \)-set.

2. The Families \( G \) and \( F \)

Let \( \mathcal{T}_1 \) denote the subdivided stars, \( \mathcal{T}_2 \) the coronas of stars, and \( \mathcal{T}_3 \) the subdivided double stars of \( \mathcal{T} \). Thus, \( \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \). Recall that \( L(T) \) denotes the set of leaves of \( T \) and \( S(T) \) the set of support vertices. Let \( X = X(T) \) consist of the leaf adjacent to the vertex of maximum degree if \( T \in \mathcal{T}_2 \) and \( T \neq P_2 \), and \( X = \emptyset \) otherwise. We also let \( H = H(T) \) consist of the center vertex if \( T \in \mathcal{T}_3 \) and \( H = \emptyset \) otherwise.

Observation 6. If \( T \) is a tree in \( \mathcal{T} \) of order at least three, then every vertex of \( B(T) \) is either a support vertex or the center vertex if \( T \in \mathcal{T}_3 \).

We define the following families of trees \( G_1, G_2, G_3 \) and \( G_4 \), and let \( \mathcal{G} = \bigcup_{i=1}^{4} G_i \), where
A tree $T$ is in $\mathcal{F}$ if it can be constructed using one of the following operations.

- **Operation $\mathcal{F}_0$:** Let $T_1$ and $T_2$ be in $\mathcal{T}$, each of order at least three. Form $T$ from $T_1 \cup T_2$ by adding an edge $xy$, where $x \in B(T_1) - H(T_1)$ and $y \in B(T_2) - H(T_2)$.

- **Operation $\mathcal{F}_1$:** Let $T_1, T_2 \in \mathcal{T}_1$. Form $T$ from $T_1 \cup T_2$ by adding an edge $xy$, where $x \in V(T_1)$, $y \in A(T_2)$.

- **Operation $\mathcal{F}_2$:** Let $T_1, T_2 \in \mathcal{T}_3$ and $T_3 \in \mathcal{T}_1$. Form $T$ from $T_1 \cup T_2$ by adding an edge $xy$, where $x \in H(T_1)$ and $y \in A(T_2)$.

- **Operation $\mathcal{F}_3$:** Let $T_1, T_2 \in \mathcal{T}$ and $T_3 \in \mathcal{T}_3$ with $T_3 \neq T_2$. Form $T$ from $T_1 \cup T_2$ by adding an edge $xy$, where $x \in B(T_1) - H(T_1)$ and $y \in A(T_2) - L(T_2)$.

- **Operation $\mathcal{F}_4$:** Let $T_1$ and $T_2$ be in $\mathcal{T}$, each of order at least four. Form $T$ from $T_1 \cup T_2$ by adding an edge $xy$, where either $x \in A(T_1) - L(T_1)$ and $y \in A(T_2) - L(T_2)$, or $x \in L(T_1) - X$, $y \in A(T_2) - L(T_2)$ and at least $T_1$ or $T_2$ is in $\mathcal{T}_1$.

- **Operation $\mathcal{F}_5$:** Let $T_1, T_2 \in \mathcal{T}_2$ and $T_3 \in \mathcal{T}$ but not both a path $P_2$. Form $T$ from $T_1 \cup T_2$ by adding a path $xzy$, where $x$ is a vertex of maximum degree in $T_1$, $y \in A(T_2) - X(T_2)$ and $z$ is a new vertex.

- **Operation $\mathcal{F}_6$:** Let $T_1, T_2 \in \mathcal{T}_1$ and $T_3 \in \mathcal{T}_3$. Form $T$ from $T_1 \cup T_2$ by adding a path $xvwzy$, where $v, w, z$ are new vertices, $x \in A(T_1)$, $y \in A(T_2)$, and at least one of $x$ and $y$ is not in $L(T_1) \cup L(T_2)$ or $x \in L(T_1)$, $y \in L(T_2)$ and $T_1 = P_3$.

- **Operation $\mathcal{F}_7$:** Let $T_1, T_2 \in \mathcal{T}_1$. Form $T$ from $T_1 \cup T_2$ by adding a path $xvwzy$, where $x \in A(T_1)$, $y \in A(T_2)$ and $v, w, z$ are new vertices.
3. Trees $T$ with $\gamma_2(T) = \gamma(T) + 2$

**Theorem 7.** A tree $T$ satisfies $\gamma_2(T) = \gamma(T) + 2$ if and only if $T \in \mathcal{G} \cup \mathcal{F}$.

**Proof.** Let $T$ be a tree with $\gamma_2(T) = \gamma(T) + 2$ and $S$ any $\gamma_2(T)$-set. For any vertex $x \in V - S$, let $S_x = N(x) \cap S$. Clearly $|S_x| \geq 2$. Since $T$ is a tree, for every pair of vertices $x, y$ in $V - S$, $|S_x \cap S_y| \leq 1$. Let $x, y$ be two adjacent vertices of $V - S$ and let $T_x, T_y$ the subtrees of $T$ obtained by removing the edge $xy$. Note that each of $T_x$ and $T_y$ has order at least three since $|S_x| \geq 2$ and $|S_y| \geq 2$. Then $S \cap V(T_x)$ and $S \cap V(T_y)$ are two $2$-dominating sets of $T_x$ and $T_y$, respectively. Hence $\gamma_2(T_x) + \gamma_2(T_y) \leq |S \cap V(T_x)| + |S \cap V(T_y)| = \gamma_2(T)$. On the other hand if $D_x$ (respectively, $D_y$) is any $\gamma(T_x)$-set (respectively, $\gamma(T_y)$-set), then $D_x \cup D_y$ is a dominating set of $T$ and so $\gamma(T) \leq \gamma(T_x) + \gamma(T_y)$. Also by Theorem 4, $\gamma_2(T_x) \geq \gamma(T_x) + 1$ and $\gamma_2(T_y) \geq \gamma(T_y) + 1$. Therefore we obtain $\gamma(T) + 2 = \gamma_2(T) \geq \gamma_2(T_x) + \gamma_2(T_y) \geq \gamma(T_x) + 1 + \gamma(T_y) + 1 \geq \gamma(T) + 2$, implying equality throughout the inequality chain, in particular $\gamma_2(T_x) = \gamma(T_x) + 1$ and $\gamma_2(T_y) = \gamma(T_y) + 1$. It follows that each of $T_x$ and $T_y$ belongs to $T - \{P_2\}$, where $x \in B(T_x)$ and $y \in B(T_y)$. If $y \in H(T_y)$, then $S(T_x) \cup S(T_y) \cup H(T_x)$ (possibly $H(T_x) = \emptyset$) is a dominating set of $T$ of size less than $\gamma_2(T) - 2$, a contradiction. Hence $y \notin H(T_y)$ and likewise $x \notin H(T_x)$. Therefore $T \in \mathcal{F}$ since it can be constructed using Operation $\mathcal{F}_0$. From now on we may assume that $V - S$ is independent.

Assume that $|S_u| \geq 4$ for some vertex $u \in V - S$. Then $\{u\} \cup S - S_u$ is a dominating set of $T$ with cardinality at most $\gamma_2(T) - 3$, a contradiction. Thus every vertex of $V - S$ has degree two or three.

Now let $x$ be a vertex of $V - S$ of degree three. Let $y \in S_x$ such that the subtrees obtained by removing the edge $xy$ are both nontrivial. If such a vertex $y$ does not exist, then $T = K_{1,3}$ that belongs to $\mathcal{G}_2$. Hence suppose that $y$ exists. Then $S \cap V(T_x)$ is a $2$-dominating set of $T_x$ and likewise $S \cap V(T_y)$ for $T_y$. Thus $\gamma_2(T_x) + \gamma_2(T_y) \leq |S \cap V(T_x)| + |S \cap V(T_y)| = \gamma_2(T)$. Moreover if $D_x$ (respectively, $D_y$) is any $\gamma(T_x)$-set (respectively, $\gamma(T_y)$-set), then $D_x \cup D_y$ is a dominating set of $T$ and so $\gamma(T) \leq \gamma(T_x) + \gamma(T_y)$. Using Theorem 4 we obtain $\gamma(T) + 2 = \gamma_2(T) \geq \gamma_2(T_x) + \gamma_2(T_y) \geq \gamma(T_x) + 1 + \gamma(T_y) + 1 \geq \gamma(T) + 2$, implying equality throughout the inequality chain, in
particular $\gamma_2(T_x) = \gamma(T_x) + 1$ and $\gamma_2(T_y) = \gamma(T_y) + 1$. It follows that each of $T_x$ and $T_y$ belongs to $T$, where $x \in B(T_x)$ and $y \in A(T_y)$. Note that since $x \in B(T_x)$, $T_x$ has order at least three. If $T_x$ and $T_y$ are in $T_1$, then $T$ can be constructed using Operation $F_1$. Thus assume that at least one of $T_x$ and $T_y$ is in $T_2 \cup T_3$, say $T_y \in T_2 \cup T_3$. Since $x \in B(T_x)$, by Observation 6, $x$ is either a support vertex or the center vertex if $T_x \in T_3$.

First assume that $x$ is a support vertex. Suppose that $y \in L(T_y)$ and let $w$ be the unique neighbor of $y$ in $T_y$. Since $T_y \in T_2 \cup T_3$ either $w \in B(T_y)$ or $w \in A(T_y)$ if $y \in X$. In addition let $z$ be the second neighbor of $w$ if $T_y \in T_3$.

Now if $T_y = P_2$, then $T_x \neq P_4$ for otherwise $T$ is a corona of a path $P_3$ and so by Theorem 4, $\gamma_2(T) = \gamma(T) + 1$, a contradiction. It follows that $T$ belongs to $G_1$. Suppose now that $T_y \neq P_2$. Then for all possibilities of $T_x$ to be in $T$ and $T_y \in T_2 \cup T_3$ with $T_y \neq P_2$, the set $S(T_x) \cup S(T_y) \cup H(T_x) \cup \{z\} - \{w\}$ (possibly $H(T_x) = \emptyset$ if $T_x \notin T_3$) is a dominating set of $T$ of size $\gamma_2(T) - 3$, a contradiction. Thus $y \in A(T_y) - L(T_y)$ and so $T$ can be constructed using Operation $F_3$.

Suppose now that $x$ is not a support vertex. Thus $x \in H(T_x)$ and hence $T_x \in T_3$. We shall show that $T_y \in T_1$. Assume that $T_y$ is in $T_2 \cup T_3$ and suppose that $y$ is not a leaf. Then since $y \in A(T_y)$, $y$ is either a neighbor of $H(T_y)$ if $T_y \in T_2$ or $y$ is the neighbor of $X(T_y)$ if $T_y \in T_2$ (in the later case $y$ is a support vertex). Anyway it can be seen that $S(T_x) \cup S(T_y) \cup Q$ is a dominating set of $T$ of size $\gamma_2(T) - 3$, where $Q = \{y\}$ if $T_y \in T_3$ and $Q = \emptyset$ otherwise. Hence $y$ is a leaf in $T_y$. Let $u$ be the unique neighbor of $y$ in $T_y$.

Clearly if $T_y = P_2$, then $S(T_x) \cup \{y\}$ is a dominating set of $T$ of size less than $\gamma_2(T) - 2$, a contradiction. Thus $T_y \neq P_2$ and so $u$ is a support vertex in $T_y$. But then $S(T_x) \cup S(T_y) \cup \{y\} \cup H(T_y) - \{u\}$ (possibly $H(T_y) = \emptyset$ if $T_y \notin T_3$) is a dominating set of $T$ of size less than $\gamma_2(T) - 2$, a contradiction too. Consequently $T_y \in T_1$ and so $T$ is constructed using Operation $F_2$.

From now on we may suppose that every vertex in $V - S$ has degree two.

Suppose now that $T$ contains a support vertex $w$ with at least two leaves. If $w \in V - S$, then by the previous assumption $\deg_T(w) = 2$ and so $T = P_3$ but then $\gamma_2(T) = \gamma(T) + 1$, a contradiction. Thus $w \in S$. Let $w'$ be any leaf neighbor of $w$ and consider the tree $T' = T - \{w\}$. Clearly $\gamma(T') = \gamma(T)$ and $\gamma_2(T') \leq \gamma_2(T) - 1$. Therefore $\gamma(T') + 1 \leq \gamma_2(T') \leq \gamma_2(T) - 1 = (\gamma(T) + 2) - 1 = \gamma(T') + 1$, implying that $\gamma_2(T') = \gamma(T') + 1$.

By Theorem 4 $T' \in T$ and $T' \neq P_2$. Hence $T \in G_2$. We may assume for the next that every support vertex is adjacent to exactly one leaf.

We now suppose that the subgraph $G[S]$ contains an edge $uv$ for which
the removing provides two nontrivial subtrees. Let $T_u$ and $T_v$ the resulting
subtrees, where $u \in V(T_u)$ and $v \in V(T_v)$. By a similar argument to that
used above we have $\gamma(T) + 2 = \gamma_2(T) \geq \gamma_2(T_u) + \gamma_2(T_v) \geq \gamma(T_u) + 1 +
\gamma(T_v) + 1 \geq \gamma(T) + 2$ and so $\gamma_2(T_u) = \gamma(T_u) + 1$, $\gamma_2(T_v) = \gamma(T_v) + 1$. Hence
each of $T_u$ and $T_v$ is in $T$, where $u \in A(T_u)$ and $v \in A(T_v)$. Also each $T_u$ and $T_v$ has order at least three for otherwise $S$ is not minimal since either
$S - \{u\}$ or $S - \{v\}$ is 2-dominating set of $T$. We also note that if $T_u \in T_2$ and $u \in X(T_u)$, then $S - \{u\}$ is 2-dominating set of $T$, a contradiction.
Thus if $T_u \in T_2$, then $u \notin X(T_u)$ and similarly if $T_v \in T_2$, then $v \notin X(T_v)$.
Now if $u$ and $v$ are both not leaves, then $|V(T_u)| \geq 4$ and $|V(T_v)| \geq 4$, and
therefore $T$ is constructed using Operation $F_4$. Assume now that $u$ and $v$
are both leaves in $T_u$ and $T_v$, respectively. If $T_u$ and $T_v$ belong to $T_1$, then $T$
is constructed by using Operation $F_1$. Thus at least one of $T_u$ and $T_v$
is in $T_2 \cup T_3$, say $T_u \in T_2 \cup T_3$. If $T_u = P_3$, then $T_v \neq P_4$ for otherwise
$T = P_7 \in T$. Consequently $T \in G_3$. Thus we assume that each of $T_u$ and $T_v$
has order at least four and recall that $u \notin X(T_u)$ and $v \notin X(T_v)$. Let
$w'$ be the support vertex of $T_u$ adjacent to $u$ and let $w'$ the support of $T_v$
adjacent to $v$. If $T_u \in T_2$, then $S(T_u) \cup S(T_v) \cup \{v\} \cup H(T_v) - (\{w', w''\}$
is a dominating set of $T$ of size less than $\gamma_2(T) - 2$, a contradiction. Thus $T_u \notin T_2$
and likewise $T_v \notin T_2$. Hence, without loss of generality, either $T_u \in T_1$
and $T_v \in T_3$ or $T_u, T_v \in T_3$. Since for both cases $T_v \in T_3$, let $w''$ be the second
neighbor of $v'$ in $T_v$. If $T_u \in T_1$ and $T_v \in T_3$, then $S(T_u) \cup S(T_v) \cup \{u, v''\} -$
$\{w', w''\}$ is a dominating set of $T$ of size $\gamma_2(T) - 3$. If $T_{u, v} \in T_3$, then
$S(T_u) \cup S(T_v) \cup H(T_u) \cup \{u, v''\} - \{w', w''\}$ is a dominating set of $T$ of size
$\gamma_2(T) - 3$. Both cases yield to a contradiction. Finally assume, without loss
of generality, that $u$ is a leaf in $T_u$ and $v$ is not a leaf in $T_v$. By examining
case by case, it can be seen that at least one of $T_u$ or $T_v$ must be in $T_1$. For
the remaining cases $T$ admits a dominating set of $T$ of size $\gamma_2(T) - 3$. Thus
$T$ can be constructed by Operation $F_4$.

Assume now that $G[S]$ contains at least one edge but each one is pendant
in $T$. Let $u \in S$ be a support and $v \in S$ its unique leaf. Let $w$ be a
vertex of $V - S$ adjacent to $u$ for which the removing provides two nontrivial
subtrees. If such a vertex does not exist, then $T$ is a corona of a star and
by Theorem 4, $\gamma_2(T) = \gamma(T) + 1$, a contradiction. Hence $w$ exists and let $r$
be the second neighbor of $w$ in $S$. Consider the nontrivial subtrees $T_r$
and $T_{u, w}$ obtained by removing $w$ (remember that $w$ has degree two in $T$). Then
$\gamma(T) + 2 = \gamma_2(T) \geq \gamma_2(T_r) + \gamma_2(T_{u, w}) \geq \gamma(T_u) + 1 + \gamma(T_v) + 1 \geq \gamma(T) + 2$ and so
$\gamma_2(T_u) = \gamma(T_u) + 1$ and $\gamma_2(T_v) = \gamma(T_v) + 1$. It follows that $T_u$ and $T_v$ belong
to $T$, where $u \in A(T_u)$ and $r \in A(T_r)$. Moreover, since $u, v \in A(T_u)$ and $u$ is a support vertex either $T_u = P_2$ or $u \in T_2$ and $u$ is the center vertex of $T_u$. Also $T_u$ and $T_r$ can not both be a path $P_2$ for otherwise $T = P_3$ and $\gamma_3(T) = \gamma(T) + 1$, a contradiction. On the other hand if $T_r \in T_2$ and $T_r \neq P_2$, then $r \notin X(T_r)$ for otherwise $S$ would also contain the support vertex of $r$ in $T_r$, say $r'$, but in this case removing the edge $rr'$ from $G[S]$ provides two nontrivial subtrees and such a case has already considered. Thus $r \in A(T_r) - X(T_r)$ and therefore $T$ can be constructed by Operation $F_5$.

Now we can assume that $S$ is independent. Since $V - S$ is an independent set in which every vertex has degree two, $T$ is the subdivision graph of a tree $T_0$. Assume that $S$ contains a vertex $x$ of degree $k \geq 2$ such that $T - N[x]$ provides $k$ nontrivial subtrees $T_1, T_2, \ldots, T_k$. Then $S \cap V(T_i)$ is a 2-dominating set of $T_i$ for every $i$ and clearly $\gamma(T) \leq 1 + \sum_{i=1}^{k} \gamma(T_i)$. Hence

$$\gamma(T) + 2 = \gamma_2(T) \geq 1 + \sum_{i=1}^{k} \gamma_2(T_i) \geq 1 + \sum_{i=1}^{k} (\gamma(T_i) + 1) \geq \gamma(T) + k \geq \gamma(T) + 2,$$

implying equality throughout the inequality chain, in particular $k = 2$, that is $\deg_T(x) = 2$, $\gamma_2(T_i) = \gamma(T_i) + 1$ for every $i = 1, 2$. Hence each of $T_1$ and $T_2$ belongs to $T$. Let $N(x) = \{x', x''\}$ and assume, without loss of generality, that $S_{x'} = \{y', x\}$ and $S_{x''} = \{y'', x\}$, where $y' \in V(T_1)$ and $y'' \in V(T_2)$. Clearly $y' \in A(T_1)$ and $y'' \in A(T_2)$. Since $S$ is independent, $T_1 \notin T_2$ and $T_2 \notin T_2$. Assume that $y'$ and $y''$ are both leaves. If $T_1, T_2 \notin T_3$, then let $y_1$ be the neighbor of $y'$ and $z_1 \neq y'$ be the neighbor of $y_1$ in $T_1$, and define similarly $y_2$ and $z_2$ in $T_2$. Then $S(T_1) \cup S(T_2) \cup \{z_1, x', x'', z_2\} - \{y_1, y_2\}$ is a dominating set of $T$ of size less than $\gamma_2(T) - 2$, a contradiction. Thus, without loss of generality, $T_1 \in T_1$ and $T_2 \in T_1 \cup T_2$. If $T_1$ has order three, then $T$ is obtained by using Operation $F_6$ (when $T_2 \in T_3$) or Operation $F_7$ (when $T_2 \in T_1$). Hence suppose that $T_1$ has order at least five. Now if $T_2 \notin T_3$, then let us use the notation of $y_1, z_1, y_2, z_2$ as have been defined above. Then $S(T_1) \cup S(T_2) \cup \{y', x'', z_2\} - \{y_1, y_2\}$ is a dominating set of $T$ of size less than $\gamma_2(T) - 2$, a contradiction. Thus $T_1 \in T_1$ and $T_2 \in T_1$, and therefore $T$ can be constructed by Operation $F_7$. For the next we will assume that at least one of $x$ and $y$ is not in $L(T_1) \cup L(T_2)$. If $T_1$ and $T_2$ are in $T_1$, then $T$ is constructed using Operation $F_7$. Hence either ($T_1 \in T_1$ and $T_2 \in T_3$) or ($T_1 \in T_3$ and $T_2 \in T_3$). In the first case $T$ is constructed using Operation $F_6$. In the later case it can be seen that $y' \in A(T_1) - L(T_1)$ and $y'' \in A(T_2) - L(T_2)$ for otherwise $T$ admits a dominating set of size less than $\gamma_2(T) - 2$, a contradiction. Thus $T$ is obtained by using Operation $F_8$. 

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Finally assume that for every vertex \( x \in S \) of degree at least two the forest \( T - N[x] \) contains a component of size one. Hence every vertex of \( S \) is either a leaf or at distance two from some leaf. Using this fact and since \( T \) is the subdivision graph of a tree \( T_0 \), it follows that every vertex of \( T_0 \) is either a support vertex or a leaf, that is \( V(T_0) = S(T_0) \cup L(T_0) \). Let \( n_0 \) be the order of \( T_0 \). Then \( |V(T)| = n = 2n_0 - 1 \) and by Theorem 3, \( \gamma_2(T) = \frac{n+1}{2} = n_0 \), implying that \( \gamma(T) = n_0 - 2 \). Suppose that a support vertex \( x \) in \( T_0 \) is adjacent to at least three other support vertices, say \( u, v \) and \( w \). Let \( u', v', w' \) be the subdivision vertices resulting by subdividing edges \( xu, xv \) and \( xw \). Clearly \( u', v', w' \in B(T) \) and \( B(T) \) is a dominating set of \( T \) of size \( n_0 - 1 \) but then \( \{x\} \cup B(T) - \{u', v', w'\} \) is a dominating set of \( T \) with cardinality \( n_0 - 3 \), a contradiction. Hence every support vertex of \( T_0 \) is adjacent to at most two other support vertices, more precisely \( T_0 \) is a caterpillar whose support vertices induce a path. If \( T_0 \) has one or two support vertices, then \( T \in \mathcal{T}_1 \) or \( T \in \mathcal{T}_3 \), respectively, and by Theorem 4, \( \gamma_2(T) = \gamma(T) + 1 \), a contradiction. Hence \( |S(T_0)| \geq 3 \). Suppose that \( |S(T_0)| \geq 5 \) and let \( u_1, u_2, \ldots, u_5 \) be five consecutive support vertices. Let \( v_i \) be the subdivision vertex resulting by subdividing the edge \( u_iu_{i+1} \), where \( 1 \leq i \leq 4 \). Then \( \{u_2, u_4\} \cup B(T) - \{v_1, v_2, v_3, v_4\} \) is a dominating set of \( T \) of size \( n_0 - 3 \), a contradiction. It follows that \( T_0 \) is a caterpillar with three or four support vertices. Hence \( T \in \mathcal{G}_4 \).

Conversely, if \( T \in \mathcal{G} \cup \mathcal{F} \), then \( T \notin \mathcal{T} \) and so by Theorem 4, \( \gamma_2(T) \geq \gamma(T) + 2 \). Equality can be checked by examining case by case the trees of \( \mathcal{G} \cup \mathcal{F} \).

Observe that any tree \( T \in \mathcal{T} \cup \mathcal{G} \cup \mathcal{F} \) has diameter at most 12, indeed the tree of larger diameter is obtained by using Operation \( \mathcal{F}_7 \) or \( \mathcal{F}_8 \). Consequently Theorems 4 and 7 imply the following corollary.

**Corollary 8.** If \( T \) is a tree of diameter at least 13, then \( \gamma_2(T) \geq \gamma(T) + 3 \).

4. Trees \( T \) with \( \gamma_0(T) = \gamma(T) + 2 \)

Hedetniemi, Hedetniemi, and Kristiansen [4] introduced several types of alliances in graphs, including the global strong offensive alliances defined as follow: A set \( S \subseteq V(G) \) is a global strong offensive alliance (abbreviated, gsoa) of \( G \) if \( |N[v] \cap S| > |N[v] - S| \) for every vertex \( v \in V(G) - S \). The
global strong offensive number $\gamma_o(G)$ is the minimum cardinality of a global strong offensive alliance of $G$.

Note if $S$ is any global strong offensive alliance of $G$, then every vertex of $V(G) - S$ has at least two neighbors in $S$. Thus $S$ is a 2-dominating set of $G$, and we obtain $\gamma_2(G) \leq \gamma_o(G)$. Using this fact, it has been observed in [1] that for every nontrivial tree $T$, $\gamma_o(T) \geq \gamma(T) + 1$ with equality if and only if $T \in T$.

Next we present a characterization of trees $T$ with $\gamma_o(T) = \gamma(T) + 2$. For this purpose let $F'$ be the subfamily of $F$ consisting of all trees constructed by performing Operation $F_0$.

**Theorem 9.** A tree $T$ satisfies $\gamma_o(T) = \gamma(T) + 2$ if and only if $T \in G \cup (F - F')$.

**Proof.** Let $T$ be a tree with $\gamma_o(T) = \gamma(T) + 2$ and $S$ any $\gamma_o(T)$-set. Clearly $\gamma_2(T) = \gamma(T) + 2$ and so $S$ is also a $\gamma_2(T)$-set. For a vertex $x \in V - S$, let $S_x = N(x) \cap S$. Then since $T$ is a tree, $|S_x \cap S_y| \leq 1$ for every pair of vertices $x, y$ in $V - S$. Assume now that $u, v$ are two adjacent vertices in $V - S$. Then since $S$ is a $\gamma_o(T)$-set, $|S_u| \geq 3$ and $|S_v| \geq 3$, and so $S \cup \{u, v\} - (S_u \cup S_v)$ is a dominating set of $T$ with cardinality at most $|S \cup \{u, v\} - (S_u \cup S_v)| \leq \gamma_0(T) - 4$, a contradiction. Thus $V - S$ is independent. Since $S$ is a $\gamma_2(T)$-set, all steps in the proof of the Theorem 7 remain valid here and therefore $T \in G \cup (F - F')$.

Conversely, every tree $T \in G \cup (F - F')$ admits a $\gamma_2(T)$-set that is also a global strong offensive alliance of $T$. Thus $\gamma(T) + 2 \leq \gamma_2(T) \leq \gamma_o(T) \leq \gamma_2(T) = \gamma(T) + 2$. Therefore $\gamma_o(T) = \gamma(T) + 2$.

**References**


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