SIGNED DOMINATION AND SIGNED DOMATIC NUMBERS OF DIGRAPHS

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Abstract

Let $D$ be a finite and simple digraph with the vertex set $V(D)$, and let $f : V(D) \to \{-1, 1\}$ be a two-valued function. If $\sum_{x \in N^-[v]} f(x) \geq 1$ for each $v \in V(D)$, where $N^-[v]$ consists of $v$ and all vertices of $D$ from which arcs go into $v$, then $f$ is a signed dominating function on $D$. The sum $f(V(D))$ is called the weight $w(f)$ of $f$. The minimum of weights $w(f)$, taken over all signed dominating functions $f$ on $D$, is the signed domination number $\gamma_S(D)$ of $D$. A set $\{f_1, f_2, \ldots, f_d\}$ of signed dominating functions on $D$ with the property that $\sum_{i=1}^d f_i(x) \leq 1$ for each $x \in V(D)$, is called a signed dominating family (of functions) on $D$. The maximum number of functions in a signed dominating family on $D$ is the signed domatic number of $D$, denoted by $d_S(D)$.

In this work we show that $4 - n \leq \gamma_S(D) \leq n$ for each digraph $D$ of order $n \geq 2$, and we characterize the digraphs attending the lower bound as well as the upper bound. Furthermore, we prove that $\gamma_S(D) + d_S(D) \leq n + 1$ for any digraph $D$ of order $n$, and we characterize the digraphs $D$ with $\gamma_S(D) + d_S(D) = n + 1$. Some of our theorems imply well-known results on the signed domination number of graphs.

Keywords: digraph, oriented graph, signed dominating function, signed domination number, signed domatic number.

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In this paper all digraphs are finite without loops or multiple arcs. A digraph without directed cycles of length 2 is an oriented graph. The vertex set and arc set of a digraph $D$ are denoted by $V(D)$ and $A(D)$, respectively. The
order $n = n(D)$ of a digraph $D$ is the number of its vertices. If $uv$ is an arc of $D$, then we also write $u \rightarrow v$, and we say that $v$ is an out-neighbor of $u$ and $u$ is an in-neighbor of $v$. If $A$ and $B$ are two disjoint vertex sets of a digraph $D$ such that $a \rightarrow b$ for each $a \in A$ and each $b \in B$, then we use the symbol $A \rightarrow B$. For a vertex $v$ of a digraph $D$, we denote the set of in-neighbors and out-neighbors of $v$ by $N^-(v) = N_D^-(v)$ and $N^+(v) = N_D^+(v)$, respectively. Furthermore, $N^-[v] = N_D^-[v] = N^-(v) \cup \{v\}$. The numbers $d_D^-(v) = d^-(v) = |N^-(v)|$ and $d_D^+(v) = d^+(v) = |N^+(v)|$ are the indegree and outdegree of $v$, respectively. The minimum indegree, maximum indegree, minimum outdegree and maximum outdegree of $D$ are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. A digraph $D$ is strongly connected if, for each pair of vertices $u$ and $v$ in $D$, there is a directed path from $u$ to $v$ in $D$. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X,v)$ is the set of arcs from $X$ to $v$. The complete digraph of order $n$ is denoted by $K_n^*$. If $X \subseteq V(D)$ and $f$ is a mapping from $V(D)$ into some set of numbers, then $f(X) = \sum_{x \in X} f(x)$.

A signed dominating function of a digraph $D$ is defined in [6] as a two-valued function $f : V(D) \rightarrow \{-1,1\}$ such that $f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq 1$ for each $v \in V(D)$. The sum $f(V(D))$ is called the weight $w(f)$ of $f$. The minimum of weights $w(f)$, taken over all signed dominating functions $f$ on $D$, is called the signed domination number of $D$, denoted by $\gamma_S(D)$. Signed domination in digraphs has been studied in [3] and [6].

A set $\{f_1,f_2,\ldots,f_d\}$ of signed dominating functions on $D$ with the property that $\sum_{i=1}^{d} f_i(x) \leq 1$ for each vertex $x \in V(D)$, is called a signed dominating family (of functions) on $D$. The maximum number of functions in a signed dominating family on $D$ is the signed domatic number of $D$, denoted by $d_S(D)$. The signed domatic number of digraphs was introduced by Sheikholeslami and Volkmann [4]. We start with a simple observation.

**Observation 1.** Let $D$ be a digraph of order $n$. If $1 \leq n \leq 2$, then $\gamma_S(D) = n$, and if $n \geq 3$, then

$$4 - n \leq \gamma_S(D) \leq n.$$ 

**Proof.** It is easy to see that $\gamma_S(D) = n$ when $1 \leq n \leq 2$. Assume now that $n \geq 3$. The upper bound $\gamma_S(D) \leq n$ is immediate. If $f$ is a signed dominating function on $D$, then the condition $n \geq 3$ implies that there are at least two distinct vertices $u$ and $v$ such that $f(u) = f(v) = 1$, and thus $\gamma_S(D) \geq 2 - (n - 2) = 4 - n$.  

$\blacksquare$
Let $\mathcal{F}$ be the family of digraphs of order $n \geq 3$ such that there exist two vertices $u$ and $v$ such $\{u, v\} \rightarrow x$ for each $x \in V(D) \setminus \{u, v\}$, the set $V(D) \setminus \{u, v\}$ is independent, and there are at most two arcs from $V(D) \setminus \{u, v\}$ to $\{u, v\}$. If there are two arcs from $V(D) \setminus \{u, v\}$ to $\{u, v\}$, then the end-vertices of these arcs are different. In addition,

- if there is no arc from $V(D) \setminus \{u, v\}$ to $\{u, v\}$, then $\{u, v\}$ is an independent set or there are one or two arcs between $u$ and $v$,
- if there is exactly one arc from $V(D) \setminus \{u, v\}$ to $\{u, v\}$, say $w \rightarrow u$, then $v \rightarrow u$,
- if there are exactly two arcs from $V(D) \setminus \{u, v\}$ to $\{u, v\}$, say $w \rightarrow u$ and $z \rightarrow v$, where $w = z$ is admissible, then $v \rightarrow u$ as well as $u \rightarrow v$.

**Theorem 2.** Let $D$ be a digraph of order $n \geq 3$. Then $\gamma_S(D) = 4 - n$ if and only if $D$ is a member of $\mathcal{F}$.

**Proof.** If $D$ is a member of $\mathcal{F}$, then it is a simple matter to verify that the function $f : V(D) \rightarrow \{-1, 1\}$ such that $f(u) = f(v) = 1$ and $f(x) = -1$ for $x \in V(D) \setminus \{u, v\}$ is a signed dominating function on $D$ of weight $4 - n$. Applying Observation 1, we obtain $\gamma_S(D) = 4 - n$.

Conversely, assume that $\gamma_S(D) = 4 - n$, and let $f$ be a signed dominating function on $D$ of weight $4 - n$. Then there exist exactly two vertices, say $u$ and $v$, such that $f(u) = f(v) = 1$ and $f(x) = -1$ for $x \in V(D) \setminus \{u, v\}$. Because of $\sum_{y \in N^{-}[x]} f(y) \geq 1$ for each $x \in V(D) \setminus \{u, v\}$, we deduce that $\{u, v\} \rightarrow x$ for every $x \in V(D) \setminus \{u, v\}$ and that $V(D) \setminus \{u, v\}$ is an independent set. If there are at least three arcs from $V(D) \setminus \{u, v\}$ to $\{u, v\}$, then $u$ or $v$, say $u$, has at least two in-neighbors in $V(D) \setminus \{u, v\}$, and we obtain the contradiction $\sum_{x \in N^{-}[u]} f(x) \leq 0$. Thus there are at most two arcs from $V(D) \setminus \{u, v\}$ to $\{u, v\}$. Now it is straightforward to verify that $D$ is a member of $\mathcal{F}$.

**Corollary 3** (Karami, Sheikholeslami, Khodar [3] 2009). If $D$ is an oriented graph of order $n \geq 3$, then $\gamma_S(D) \geq 4 - n$ with equality if and only if there exist two vertices $u$ and $v$ such $\{u, v\} \rightarrow x$ for each $x \in V(D) \setminus \{u, v\}$, the set $V(D) \setminus \{u, v\}$ is independent, and $\{u, v\}$ is independent or there is exactly one arc between $u$ and $v$.

**Corollary 4.** If $D$ is a strongly connected digraph of order $n \geq 5$, then $\gamma_S(D) \geq 6 - n$. 
Let $H$ be the digraph of order $n \geq 5$ with vertex set $V(D) = \{u, v, w, x_1, x_2, \ldots, x_{n-3}\}$ such that $\{u, v, w\} \rightarrow \{x_1, x_2, \ldots, x_{n-3}\}$, $x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{n-3} \rightarrow w$ and $w \rightarrow v \rightarrow u \rightarrow w$. Then $H$ is strongly connected, and the function $f : V(H) \rightarrow \{-1, 1\}$ such that $f(u) = f(v) = f(w) = 1$ and $f(x_i) = -1$ for $1 \leq i \leq n - 3$ is a signed dominating function on $D$ of weight $6 - n$. Therefore the bound given in Corollary 4 is best possible.

Let $Q$ be the digraph of order $n = 4$ with vertex set $V(D) = \{u, v, x_1, x_2\}$ such that $\{u, v\} \rightarrow \{x_1, x_2\}$, $x_1 \rightarrow u$, $x_2 \rightarrow v$, $u \rightarrow v$ and $v \rightarrow u$. Then $Q$ is strongly connected, and the function $f : V(Q) \rightarrow \{-1, 1\}$ such that $f(u) = f(v) = 1$ and $f(x_1) = f(x_2) = -1$ is a signed dominating function on $Q$ of weight $0$. This example demonstrates that Corollary 4 does not hold for $n = 4$.

**Theorem 5.** If $D$ is a strongly connected oriented graph of order $n \geq 7$, then $\gamma_S(D) \geq 8 - n$, and this bound is sharp.

**Proof.** According to Corollary 4, we have $\gamma_S(D) \geq 6 - n$. Suppose to the contrary that $\gamma_S(D) = 6 - n$, and let $f$ be a signed dominating function on $D$ of weight $6 - n$. Then there exist exactly three vertices, say $u, v$ and $w$, such that $f(u) = f(v) = f(w) = 1$ and $f(x) = -1$ for $x \in V(D) \setminus \{u, v, w\}$. Because of $\sum_{y \in N^{-}[x]} f(y) \geq 1$ for each $x \in V(D) \setminus \{u, v, w\}$, each such vertex has at least two in-neighbors in $\{u, v, w\}$. Let $V(D) \setminus \{u, v, w\} = \{x_1, x_2, \ldots, x_{n-3}\}$.

First we show that $V(D) \setminus \{u, v, w\}$ is an independent set. Suppose to the contrary that there exists an arc, say $x_1x_2$, in $V(D) \setminus \{u, v, w\}$. Then $\{u, v, w\} \rightarrow x_2$, and since $D$ is a strongly connected oriented graph, $x_2$ dominates a further vertex, say $x_3$, in $V(D) \setminus \{u, v, w\}$. Thus $\{u, v, w\} \rightarrow x_3$, and since $D$ is a strongly connected oriented graph, $x_3$ dominates a further vertex of $V(D) \setminus \{u, v, w\}$. If we continue this process we arrive at a directed cycle $C_1$, say $C_1 = x_1x_2\ldots x_kx_1$ with $k \geq 3$. This implies that $\{u, v, w\} \rightarrow V(C_1)$. Since $D$ is an oriented graph, there is no arc from $C_1$ to $\{u, v, w\}$. If $k = n - 3$, then $D$ is not strongly connected, a contradiction. Otherwise, as $D$ is strongly connected, there exists an arc $az$ from $C_1$ to $V(D) \setminus (V(C_1) \cup \{u, v, w\})$. This implies $\{u, v, w\} \rightarrow z$. As above the vertex $z$ is contained in a cycle $C_2$ such that $V(C_2) \subseteq (V(D) \setminus (V(C_1) \cup \{u, v, w\}))$. But this leads to the contradiction $\sum_{x \in N^{-}[z]} f(x) \leq 0$, and thus $V(D) \setminus \{u, v, w\}$ is an independent set.

Since $D$ is strongly connected, we deduce that each vertex of $V(D) \setminus \{u, v, w\}$ has an out-neighbor in $\{u, v, w\}$. The hypothesis $n \geq 7$ implies
that at least one vertex in \( \{u, v, w\} \), say \( u \), has at least two in-neighbors in \( V(D) \setminus \{u, v, w\} \). If \( u \) has at least three in-neighbors in \( V(D) \setminus \{u, v, w\} \), then we obtain the contradiction \( \sum_{x \in N^-[u]} f(x) \leq 0 \). If \( u \) has exactly two in-neighbors in \( V(D) \setminus \{u, v, w\} \), then it follows that \( \{v, w\} \to u \). If \( v \) or \( w \), say \( v \), has two in-neighbors in \( V(D) \setminus \{u, v, w\} \), then it follows that \( \{u, w\} \to v \), a contradiction to the fact that \( D \) is an oriented graph. Finally, if \( v \) and \( w \) have exactly one in-neighbor in \( V(D) \setminus \{u, v, w\} \), then \( w \to v \), and we obtain the contradiction \( u \to w \) or \( v \to w \). This contradiction implies that \( \gamma_S(D) \geq 8 - n \).

In order to prove that this bound is sharp, let \( H \) be the digraph of order \( n \geq 7 \) with vertex set \( V(H) = \{u, v, w, z, x_1, x_2, \ldots, x_{n-4}\} \) such that \( \{v, w, z\} \to \{x_1, x_2, \ldots, x_{n-4}\}, x_1 \to u \to \{x_2, x_3, \ldots, x_{n-4}\}, x_1 \to x_2 \to \cdots \to x_{n-4} \to x_1 \) and \( u \to v \to w \to z \to u \). Then \( H \) is a strongly connected oriented graph, and the function \( f : V(H) \to \{-1, 1\} \) such that \( f(u) = f(v) = f(w) = f(z) = 1 \) and \( f(x_i) = -1 \) for \( 1 \leq i \leq n-4 \) is a signed dominating function on \( H \) of weight \( 8 - n \). Therefore \( \gamma_S(H) \leq 8 - n \), and thus \( \gamma_S(H) = 8 - n \).

Let \( Q \) be the digraph of order \( n = 6 \) with vertex set \( V(Q) = \{u, v, w, x_1, x_2, x_3\} \) such that \( u \to \{x_2, x_3\}, v \to \{x_1, x_3\}, w \to \{x_1, x_2\}, x_1 \to u, x_2 \to v, x_3 \to w \) and \( u \to v \to w \to u \). Then \( Q \) is a strongly connected oriented graph, and the function \( f : V(Q) \to \{-1, 1\} \) such that \( f(u) = f(v) = f(w) = 1 \) and \( f(x_1) = f(x_2) = f(x_3) = -1 \) is a signed dominating function on \( Q \) of weight \( 0 \). This example demonstrates that Theorem 5 does not hold for \( n = 6 \).

**Theorem 6.** Let \( r \geq 0 \) be an integer, and let \( D \) be an oriented graph of order \( n \) such that \( d^-(x) = r \) for every vertex \( x \in V(D) \). Then

\[
\gamma_S(D) \geq 2r + 2 - n \text{ if } r \text{ is even}
\]

and

\[
\gamma_S(D) \geq 2r + 4 - n \text{ if } r \text{ is odd}.
\]

**Proof.** Let \( f \) be an arbitrary signed dominating function on \( D \), and let \( V^+ \) be the set of vertices with \( f(x) = 1 \) for \( x \in V^+ \) and \( V^- = V(D) \setminus V^+ \). Furthermore, define \( |V^+| = t \).

First, let \( r = 2k \) be even. Because of \( \sum_{x \in N^-[u]} f(x) \geq 1 \) for each vertex \( u \), every vertex \( x \in V^+ \) has at most \( k \) in-neighbors in \( V^- \). It follows that
2kt = \sum_{x \in V^+} d^-(x) \leq kt + \frac{t(t-1)}{2}

and thus \( t \geq 2k + 1 \). Since \( f \) was chosen arbitrary, this implies the desired bound \( \gamma_S(D) \geq 2k + 1 - (n - (2k + 1)) = 4k + 2 - n = 2r + 2 - n \).

Second, let \( r = 2k - 1 \) be odd. Because of \( \sum_{x \in N[u]} f(x) \geq 1 \) for each vertex \( u \), every vertex \( x \in V^+ \) has at most \( k - 1 \) in-neighbors in \( V^- \). It follows that

\[
(2k-1)t = \sum_{x \in V^+} d^-(x) \leq t(k-1) + \frac{t(t-1)}{2}
\]

and thus \( t \geq 2k + 1 \). This implies that \( \gamma_S(D) \geq 2k + 1 - (n - (2k + 1)) = 4k + 2 - n = 2r + 2 - n \), and the proof is complete. \( \blacksquare \)

**Theorem 7.** If \( D \) is a digraph of order \( n \), then

\[
\gamma_S(D) \geq \frac{\delta^+ + 2 - \Delta^+}{\delta^+ + 2 + \Delta^+} \cdot n.
\]

**Proof.** Let \( f \) be an arbitrary signed dominating function on \( D \), and let \( V^+ \) be the set of vertices with \( f(x) = 1 \) for \( x \in V^+ \) and \( V^- = V(D) \setminus V^+ \). Then

\[
n \leq \sum_{x \in V(D)} f(N^-[x]) = \sum_{x \in V(D)} (d^+(x) + 1)f(x)
\]

\[
= \sum_{x \in V^+} (d^+(x) + 1) - \sum_{x \in V^-} (d^+(x) + 1)
\]

\[
\leq |V^+|(|\Delta^+ + 1| - |V^-||\delta^+ + 1|)
\]

\[
= |V^+|(|\Delta^+ + \delta^+ + 2| - n(\delta^+ + 1)).
\]

This implies

\[
|V^+| \geq \frac{n(\delta^+ + 2)}{\delta^+ + 2 + \Delta^+},
\]

and hence we obtain the desired bound as follows

\[
\gamma_S(D) \geq |V^+| - |V^-| = 2|V^+| - n
\]
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≥ \frac{2n(\delta^+ + 2)}{\delta^+ + 2 + \Delta^+} - n

= \frac{\delta^+ + 2 - \Delta^+}{\delta^+ + 2 + \Delta^+} \cdot n.

**Corollary 8.** If \( D \) is a digraph of order \( n \) such that \( d^+(x) = k \) for all \( x \in V(D) \), then

\[ \gamma_S(D) \geq \frac{n}{k+1}. \]

**Corollary 9** (Karami, Sheikholeslami, Khodar [3] 2009). If \( D \) is a digraph of order \( n \) such that \( d^-(x) = d^+(x) = k \) for all \( x \in V(D) \), then

\[ \gamma_S(D) \geq \frac{n}{k+1}. \]

If \( f \) is a signed dominating function on \( D \), and \( d^-(v) \) is odd, then it follows that \( f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq 2 \). Using this inequality, we obtain the next result analogously to the proof of Theorem 7.

**Theorem 10.** If \( D \) is a digraph of order \( n \) such that \( d^-(v) \) is odd for all \( v \in V(D) \), then

\[ \gamma_S(D) \geq \frac{\delta^+ + 4 - \Delta^+}{\delta^+ + 2 + \Delta^+} \cdot n. \]

**Corollary 11.** Let \( D \) be a digraph of order \( n \) such that \( d^-(x) = d^+(x) = k \) for all \( x \in V(D) \). If \( k \) is odd, then

\[ \gamma_S(D) \geq \frac{2n}{k+1}. \]

**Theorem 12.** If \( D \) is a digraph of order \( n \), then

\[ \gamma_S(D) \geq \frac{n + |A(D)| - n\Delta^+}{\Delta^+ + 1}. \]

**Proof.** Let \( f \) be an arbitrary signed dominating function on \( D \), and let \( V^+ \) be the set of vertices with \( f(x) = 1 \) for \( x \in V^+ \) and \( V^- = V(D) \setminus V^+ \). Then

\[ n \leq \sum_{x \in V(D)} f(N^-[x]) = \sum_{x \in V(D)} (d^+(x) + 1)f(x) \]
= \sum_{x \in V^+} (d^+(x) + 1) - \sum_{x \in V^-} (d^+(x) + 1) \\
= |V^+| - |V^-| + \sum_{x \in V^+} d^+(x) - \sum_{x \in V^-} d^+(x) \\
= 2|V^+| - n + 2 \sum_{x \in V^+} d^+(x) - \sum_{x \in V(D)} d^+(x) \\
\leq 2|V^+| - n + 2|V^+| \Delta^+ - |A(D)| \\
= 2|V^+| (\Delta^+ + 1) - n - |A(D)|.

This implies

\[ |V^+| \geq \frac{2n + |AD|}{2(\Delta^+ + 1)}. \]

and hence we obtain the desired bound as follows

\[ \gamma_S(D) \geq |V^+| - |V^-| = 2|V^+| - n \]

\[ \geq \frac{2n + |A(D)|}{\Delta^+ + 1} - n \]

\[ = \frac{n + |A(D)| - n\Delta^+}{\Delta^+ + 1}. \]

Theorem 12 also implies Corollary 8 immediately. In the special case that \( d^-(v) \) is odd for all \( v \in V(D) \), we obtain \( \gamma_S(D) \geq (2n + |A(D)| - n\Delta^+)/ (\Delta^+ + 1) \) instead of the bound in Theorem 12.

The signed dominating function of a graph \( G \) is defined in [1] as a function \( f : V(G) \to \{-1, 1\} \) such that \( \sum_{x \in N_G[v]} f(x) \geq 1 \) for all \( v \in V(G) \). The sum \( \sum_{x \in V(G)} f(x) \) is the weight \( w(f) \) of \( f \). The minimum of weights \( w(f) \), taken over all signed dominating functions \( f \) on \( G \) is called the signed domination number of \( G \), denoted by \( \gamma_S(G) \).

The associated digraph \( D(G) \) of a graph \( G \) is the digraph obtained when each edge \( e \) of \( G \) is replaced by two oppositely oriented arcs with the same ends as \( e \). Since \( N^{-}_D(v) = N_G(v) \) for each vertex \( v \in V(G) = V(D(G)) \), the following useful observation is valid.
Observation 13. If \( D(G) \) is the associated digraph of a graph \( G \), then \( \gamma_S(D(G)) = \gamma_S(G) \).

There are a lot of interesting applications of Observation 13, as for example the following three results.

**Corollary 14** (Zhang, Xu, Li, Liu [7] 1999). If \( G \) is a graph of order \( n \), maximum degree \( \Delta(G) \) and minimum degree \( \delta(G) \), then

\[
\gamma_S(G) \geq \frac{\delta(G) + 2 - \Delta(G)}{\delta(G) + 2 + \Delta(G)} \cdot n.
\]

**Proof.** Since \( \delta(G) = \delta^+(D(G)) \), \( \Delta(G) = \Delta^+(D(G)) \) and \( n = n(D(G)) \), it follows from Theorem 7 and Observation 13 that

\[
\gamma_S(G) = \gamma_S(D(G)) \geq \frac{\delta^+(D(G)) + 2 - \Delta^+(D(G))}{\delta^+(D(G)) + 2 + \Delta^+(D(G))} \cdot n = \frac{\delta(G) + 2 - \Delta(G)}{\delta(G) + 2 + \Delta(G)} \cdot n.
\]

**Corollary 15** (Dunbar, Hedetniemi, Henning, Slater [1] 1995). If \( G \) is a \( k \)-regular graph of order \( n \), then \( \gamma_S(G) \geq n/(k + 1) \).

**Corollary 16** (Henning, Slater [2] 1996). For every \( k \)-regular graph \( G \) of order \( n \) with \( k \) odd, \( \gamma_S(G) \geq 2n/(k + 1) \).

**Proof.** Since \( k \) is odd and \( d_G(x) = d^-_{D(G)}(x) = d^+_{D(G)}(x) = k \) for all \( x \in V(G) \) and \( n = n(D(G)) \), it follows from Corollary 11 and Observation 13 that

\[
\gamma_S(G) = \gamma_S(D(G)) \geq \frac{2n(D(G))}{k + 1} = \frac{2n(G)}{k + 1}.
\]

**Theorem 17.** If \( D \) is a digraph of order \( n \), then

\[
\gamma_S(D) \geq n \left( \frac{2 \left\lfloor \frac{\delta^-(D)}{2} \right\rfloor + 1 - \Delta^+(D)}{\Delta^+(D) + 1} \right).
\]

**Proof.** Let \( f \) be a signed dominating function on \( D \) such that \( w(f) = \gamma_S(D) \), and let \( V^+ \) be the set of vertices with \( f(x) = 1 \) for \( x \in V^+ \) and \( V^- = V(D) \setminus V^+ \). In addition, let \( s \) be the number of arcs from \( V^+ \) to \( V^- \).
The condition \( f(N^-[x]) \geq 1 \) implies that \( |E(V^+, x)| \geq |E(V^-, x)| \) for \( x \in V^+ \) and \( |E(V^+, x)| \geq |E(V^-, x)| + 2 \) for \( x \in V^- \). Thus we obtain

\[
\delta^-(D) \leq d^-(x) = |E(V^+, x)| + |E(V^-, x)| \leq 2|E(V^+, x)| - 2
\]

and so \( |E(V^+, x)| \geq \left\lceil \frac{\delta^-(D)+2}{2} \right\rceil \) for each vertex \( x \in V^- \). Hence we deduce that

(1) \[
s = \sum_{x \in V^-} |E(V^+, x)| \geq \sum_{x \in V^-} \left\lceil \frac{\delta^-(D)+2}{2} \right\rceil = |V^-| \left\lceil \frac{\delta^-(D)+2}{2} \right\rceil.
\]

Since \( |E(V^+, x)| \geq \left\lceil \frac{\delta^-(D)}{2} \right\rceil \) for \( x \in V^+ \), it follows that

\[
|E(D[V^+])| = \sum_{y \in V^+} |E(V^+, y)| \geq |V^+| \left\lceil \frac{\delta^-(D)}{2} \right\rceil.
\]

This implies that

(2) \[
s = \sum_{y \in V^+} d^+(y) - |E(D[V^+])| \leq \sum_{y \in V^+} d^+(y) - |V^+| \left\lceil \frac{\delta^-(D)}{2} \right\rceil \leq |V^+| \Delta^+(D) - |V^+| \left\lceil \frac{\delta^-(D)}{2} \right\rceil.
\]

Inequalities (1) and (2) lead to

\[
|V^-| \leq \frac{|V^+| \Delta^+(D) - |V^+| \left\lceil \frac{\delta^-(D)}{2} \right\rceil}{\left\lceil \frac{\delta^-(D)+2}{2} \right\rceil}.
\]

Since \( \gamma_S(D) = |V^+| - |V^-| \) and \( n = |V^+| + |V^-| \), it follows from the last inequality that

\[
\gamma_S(D) \geq |V^+| - \frac{|V^+| \Delta^+(D) - |V^+| \left\lceil \frac{\delta^-(D)}{2} \right\rceil}{\left\lceil \frac{\delta^-(D)+2}{2} \right\rceil}
\]
\[ = \left( \frac{n + \gamma_S(D)}{2} \right) 2 \left\lceil \frac{\Delta^-(D)}{2} \right\rceil + 1 - \Delta^+(D) \]

and this yields to the desired bound. \hfill \blacksquare

Note that Observation 13 and Theorem 17 also imply Corollaries 15 and 16 immediately.

**Theorem 18.** For any digraph \( D \), \( \gamma_S(D) = n(D) \) if and only if every vertex has either indegree less or equal one or is an in-neighbor of a vertex of indegree one.

**Proof.** Assume that every vertex has either indegree less or equal one or is an in-neighbor of a vertex of indegree one. Let \( f \) be an arbitrary signed dominating function on \( D \). If \( v \) is vertex such that \( d^-(v) \leq 1 \), then the definition of the signed dominating function implies that \( f(v) = 1 \). If \( v \) is an in-neighbor of a vertex \( y \) such that \( d^-(y) = 1 \), then the condition \( \sum_{x \in N^{-}[y]} f(x) \geq 1 \) leads to \( f(v) = 1 \). Hence \( f(v) = 1 \) for each \( v \in V(D) \) and we deduce that \( \gamma_S(D) = n(D) \).

The necessity follows from the observation that if we have a vertex \( v \) that is neither of indegree less or equal one nor an in-neighbor of a vertex of indegree one, then we can assign the value -1 to \( v \) and the value 1 to each other vertex to produce a signed dominating function on \( D \) of weight \( n(D) - 2 \).

The following known results are useful for the proof of our last theorem.

**Theorem A** (Sheikholeslami, Volkmann [4]). For any digraph \( D \),

\[ \gamma_S(D) \cdot d_S(D) \leq n(D). \]

**Theorem B** (Sheikholeslami, Volkmann [4]). For any digraph \( D \),

\[ 1 \leq d_S(D) \leq \delta^-(D) + 1. \]

**Theorem C** (Sheikholeslami, Volkmann [4]). The signed domatic number of a digraph is an odd integer.

**Theorem D** (Sheikholeslami, Volkmann [4] and Volkmann, Zelinka [5]). Let \( K^*_n \) be the complete digraph of order \( n \). Then \( d_S(K^*_n) = n \) if \( n \) is odd,
and if \( n = 2p \) is even, then \( d_S(K_n^*) = p \) if \( p \) is odd and \( d_S(K_n^*) = p - 1 \) if \( p \) is even.

**Theorem 19.** If \( D \) is a digraph of order \( n \), then

\[
\gamma_S(D) + d_S(D) \leq n + 1
\]

with equality if and only if \( n \) is odd and \( D = K_n^* \) or every vertex of \( D \) has either indegree less or equal one or is an in-neighbor of a vertex of indegree one.

**Proof.** According to Theorem A, we obtain

\[
\gamma_S(D) + d_S(D) \leq \frac{n}{d_S(D)} + d_S(D).
\]

Using the fact that \( g(x) = x + n/x \) is decreasing for \( 1 \leq x \leq \sqrt{n} \) and increasing for \( \sqrt{n} \leq x \leq n \), this inequality leads to (3) immediately.

If \( n \) is odd and \( D = K_n^* \), then \( \gamma_S(D) = 1 \) and Theorem D implies \( d_S(D) = n \), and we obtain equality in (3). If every vertex of \( D \) has either indegree less or equal one or is an in-neighbor of a vertex of indegree one, then Theorems B, C and 18 yield that \( \gamma_S(D) = n \) and \( d_S(D) = 1 \), and so we have equality in (3) too.

Conversely, assume that \( D \) is neither complete of odd order nor that every vertex of \( D \) has either indegree less or equal one or is an in-neighbor of a vertex of indegree one. First we note that every digraph of order \( 1 \leq n \leq 3 \) is complete of odd order or every vertex of \( D \) has either indegree less or equal one or is an in-neighbor of a vertex of indegree one, and hence \( \gamma_S(D) + d_S(D) = n + 1 \) for \( n \in \{1, 2, 3\} \).

Assume now that \( n \geq 4 \). If \( D \) is not complete, then \( \delta^-(D) \leq n - 2 \), and thus Theorem B leads to \( d_S(D) \leq n - 1 \). If \( D \) is complete and \( n \) is even, then Theorem D implies \( d_S(D) \leq n/2 \leq n - 1 \). Thus, in view of Theorem 18, we observe that \( d_S(D) \leq n - 1 \) and \( \gamma_S(G) \leq n - 1 \) if \( D \) is neither complete of odd order nor that every vertex of \( D \) has either indegree less or equal one or is an in-neighbor of a vertex of indegree one. If \( d_S(D) = 1 \), then we deduce that \( \gamma_S(D) + d_S(D) \leq 1 + n - 1 = n \). If \( d_S(D) \geq 2 \), then as above and since \( n \geq 4 \), we obtain

\[
\gamma_S(D) + d_S(D) \leq \frac{n}{d_S(D)} + d_S(D) \leq \max \left\{ \frac{n}{2} + 2, \frac{n}{n - 1} + n - 1 \right\} < n + 1.
\]
Hence the equality $\gamma_S(D) + d_S(D) = n + 1$ is impossible in this case, and the proof of Theorem 19 is complete.

Note that the inequality (3) was proved in [4], however, the characterization of the digraphs $D$ with $\gamma_S(D) + d_S(D) = n + 1$ is new.

References


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