GRAPHS WITH EQUAL DOMINATION AND 2-DISTANCE DOMINATION NUMBERS

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Abstract

Let $G = (V, E)$ be a graph. The distance between two vertices $u$ and $v$ in a connected graph $G$ is the length of the shortest $(u - v)$ path in $G$. A set $D \subseteq V(G)$ is a dominating set if every vertex of $G$ is at distance at most 1 from an element of $D$. The domination number of $G$ is the minimum cardinality of a dominating set of $G$. A set $D \subseteq V(G)$ is a 2-distance dominating set if every vertex of $G$ is at distance at most 2 from an element of $D$. The 2-distance domination number of $G$ is the minimum cardinality of a 2-distance dominating set of $G$. We characterize all trees and all unicyclic graphs with equal domination and 2-distance domination numbers.

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1. Definitions

Here we consider simple undirected graphs $G = (V, E)$ with $|V| = n(G)$. The distance $d_G(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $(u - v)$ path in $G$. If $D$ is a set and $u \in V(G)$, then $d_G(u, D) = \min\{d_G(u, v) : v \in D\}$. The $k$-neighbourhood $N^k_G[v]$ of a vertex $v \in V(G)$ is the set of all vertices at distance at most $k$ from $v$. For a set $D \subseteq V$, the $k$-neighbourhood $N^k_G[D]$ is defined to be $\bigcup_{v \in D} N^k_G[v]$. A
subset $D$ of $V$ is \textit{k-distance dominating} in $G$ if every vertex of $V(G) - D$ is at distance at most $k$ from at least one vertex of $D$. Let $\gamma^k(G)$ be the minimum cardinality of a $k$-distance dominating set of $G$. This kind of domination was defined by Borowiecki and Kuzak [1]. Note that the 1-distance domination number is the \textit{domination number}, denoted $\gamma(G)$.

The degree of a vertex $v$ is $d_G(v) = |N_G[v]|$ and a vertex of degree 1 is called a \textit{leaf}. A vertex which is a neighbour of a leaf is called a \textit{support vertex}. Denote by $S(G)$ the set of all support vertices of $G$. If a support vertex is adjacent to more than one leaf, then we call it a \textit{strong support vertex}. We denote a path on $n$ vertices by $P_n = (v_0, \ldots, v_{n-1})$ and the cycle on $n$ vertices by $C_n$. For example, $P_2$ contains two leaves and two support vertices. For any unexplained terms and symbols see [2].

In this paper we study trees and unicyclic graphs for which the domination number and the 2-distance domination number are the same.

2. General results

First we give some general results for graphs with equal domination and 2-distance domination numbers. Obviously, for any graph $G$ if $\gamma(G) = 1$, then $\gamma^2(G) = 1$ and thus $\gamma(G) = \gamma^2(G)$. We start with a necessary condition for a graph $G$ with $1 < \gamma(G) = \gamma^2(G)$. A set $D \subseteq V(G)$ is a \textit{2-packing} in $G$ if $d_G(u, v) \geq 3$ for every $u, v \in D$.

Proposition 1. If $G$ is a connected graph with $\gamma(G) = \gamma^2(G)$ and $\gamma(G) > 1$, then every minimum dominating set of $G$ is a 2-packing of $G$.

\textbf{Proof.} Suppose $D$ is a minimum dominating set of $G$ such that $|D| \geq 2$ and $D$ is not a 2-packing. Then there exist $u, v \in D$ in $G$ such that $d_G(u, v) \leq 2$. Denote by $x$ a vertex which belongs to $N_G[u] \cap N_G[v]$ (if $u$ and $v$ are adjacent, then possibly $x = u$ or $x = v$) and let $D' = (D - \{u, v\}) \cup \{x\}$. Then $N_G[u] \subseteq N_G^2[x]$ and $N_G[v] \subseteq N_G^2[x]$. Hence $D'$ is a 2-distance dominating set of $G$ of smaller cardinality than $\gamma(G)$, a contradiction. \hfill $\blacksquare$

The condition in Proposition 1 it not sufficient. Consider, for example the cycle $C_9$. Next result gives a sufficient condition for a graph $G$ to have equal domination and 2-distance domination numbers.

Proposition 2. Let $G$ be the graph obtained from a graph $H$ and $n(H)$ copies of $P_2$, where the $i$th vertex of $H$ is adjacent to exactly one vertex of the $i$th copy of $P_2$. Then $\gamma(G) = \gamma^2(G)$.
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Proof. Let $G$ be the graph obtained from a graph $H$ and $n(H)$ copies of $P_2$, where the $i$th vertex of $H$ is adjacent to exactly one vertex of the $i$th copy of $P_2$. Denote by $D$ a $\gamma^2(G)$-set. Observe that the distance between any two leaves adjacent to two different support vertices in $G$ is greater than or equal to 5. For this reason, if $u$ and $v$ are two leaves adjacent to two different support vertices, then $u$ and $v$ cannot be 2-dominated by the same element of $D$. This implies that $\gamma^2(G) \geq |S(G)|$. Since $\gamma^2(G) \leq \gamma(G)$, it follows that $\gamma(G) = \gamma^2(G)$.

3. Trees

In what follows, we constructively characterize all trees $T$ for which $\gamma(T) = \gamma^2(T)$.

Let $\mathcal{T}$ be the family of all trees $T$ that can be obtained from sequence $T_1, \ldots, T_j$ ($j \geq 1$) of trees such that $T_1$ is the path $P_2$ and $T = T_j$, and, if $j > 1$, then $T_{i+1}$ can be obtained recursively from $T_i$ by the operation $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$:

- **Operation $\mathcal{T}_1$.** The tree $T_{i+1}$ is obtained from $T_i$ by adding a vertex $x_1$ and the edge $x_1y$ where $y \in V(T_i)$ is a support vertex of $T_i$.

- **Operation $\mathcal{T}_2$.** The tree $T_{i+1}$ is obtained from $T_i$ by adding a path $(x_1, x_2, x_3)$ and the edge $x_1y$ where $y \in V(T_i)$ is neither a leaf nor a support vertex in $T_i$.

- **Operation $\mathcal{T}_3$.** The tree $T_{i+1}$ is obtained from $T_i$ by adding a path $(x_1, x_2, x_3, x_4)$ and the edge $x_1y$ where $y \in V(T_i)$ is a support vertex in $T_i$.

Additionally, let $P_1$ belong to $\mathcal{T}$.

The following observation follows immediately from the way in which each tree in the family $\mathcal{T}$ is constructed.

**Observation 3.** If a tree $T$ belonging to the family $\mathcal{T}$ has at least 2 vertices, then:

1. If $u, v \in S(T)$, then $d_T(u, v) \geq 3$, that is, if $u, v \in S(T)$, then $S(T)$ is a 2-packing in $T$;
2. If $u \in V(T)$, then $|N_T[u] \cap S(T)| = 1$;
3. $S(T)$ is a minimum dominating set of $T$.

We show first that each tree $T$ belonging to the family $\mathcal{T}$ is a tree with $\gamma(T) = \gamma^2(T)$. To this aim we prove the following lemma.
Lemma 4. If a tree $T$ of order at least 2 belongs to the family $\mathcal{T}$, then $\gamma^2(T) = |S(T)|$.

Proof. Let $T$ be a tree belonging to the family $\mathcal{T}$ and let $D$ be a $\gamma^2(T)$-set. Since $S(T)$ is a 2-packing in $T$, the distance between any two leaves adjacent to different support vertices is greater than or equal to 5. For this reason, if $u$ and $v$ are two leaves adjacent to different support vertices in $T$, then $u$ and $v$ cannot be 2-distance dominated by the same element of $D$. This implies that $|D| \geq |S|$. On the other hand, since $S(T)$ is a dominating set of $T$, it is also a 2-distance dominating set of $T$. We conclude that $\gamma^2(T) = |S(T)|$. 

By Lemma 4 and Observation 3 we obtain immediately.

Corollary 5. If a tree $T$ belongs to the family $\mathcal{T}$, then $\gamma(T) = \gamma^2(T)$.

Before we prove our next Lemma, observe that for any tree $T$ with at least 3 vertices, $\gamma(T) \geq |S(T)|$.

Lemma 6. If $T$ is a tree with $\gamma^2(T) = \gamma(T)$, then $T$ belongs to the family $\mathcal{T}$.

Proof. Let $T$ be a tree with $\gamma^2(T) = \gamma(T)$. Let $(v_0, v_1, \ldots, v_k)$ be a longest path in $T$. If $k \in \{1, 2\}$, then $T$ is $P_1$ or a star $K_{1,p}$, for a positive integer $p$, and clearly $T$ is in $\mathcal{T}$.

If $k \in \{3, 4\}$, then $\gamma^2(T) = 1$, but $\gamma(T) > 1$. For this reason now we assume $k \geq 5$. We proceed by induction on the number $n(T)$ of vertices of a tree $T$ with $\gamma^2(T) = \gamma(T)$. If $n(T) = 6$, then $T = P_6$ and $T$ belongs to the family $\mathcal{T}$. (Observe that $P_6$ may be obtained from $P_2$ by operation $\mathcal{J}_3$). Now let $T$ be a tree with $\gamma^2(T) = \gamma(T)$ and $n(T) \geq 7$, and assume that each tree $T'$ with $n(T') < n(T)$, $k \geq 5$ and $\gamma^2(T') = \gamma(T')$ belongs to the family $\mathcal{T}$.

If there exists $v \in S(T)$ such that $v$ is adjacent to at least two leaves, say $x_1$ and $x_2$, then clearly $\gamma(T') = \gamma(T)$ and $\gamma^2(T') = \gamma^2(T)$, where $T' = T-x_1$. Thus, $\gamma^2(T') = \gamma(T')$ and by the induction, $T'$ belongs to the family $\mathcal{T}$. Moreover, $T$ may be obtained from $T'$ by operation $\mathcal{J}_1$ and we conclude that $T$ also belongs to the family $\mathcal{T}$.

Now assume that each support vertex of $T$ is adjacent to exactly one leaf. For this reason $d_T(v_1) = 2$. If $d_T(v_2) > 2$, then $v_2$ is adjacent to a leaf or $|N_T(v_2) \cap S(T)| \geq 2$. In both cases $v_2$ 2-distance dominates all support vertices and leaves at distance at most 2 from $v_2$, while $\gamma(T) \geq |S(T)|$. Hence $\gamma(T) > \gamma^2(T)$, which is impossible. Thus, $d_T(v_2) = 2$. 


Observe that either \( v_0 \) or \( v_1 \) is in every minimum dominating set of \( T \). Assume \( d_T(v_3) > 2 \). If \( v_3 \) belongs to some minimum dominating set of \( T \), say \( D \), then \( (D \cup \{v_2\}) - \{v_0, v_1, v_3\} \) is a 2-distance dominating set of \( T \) of cardinality smaller than \( \gamma(T) \), which is impossible. Hence \( v_3 \) does not belong to any minimum dominating set of \( T \) and this reason together with \( n(T) \geq 7 \) imply that \( v_3 \) is not a support vertex of \( T \). Denote \( T' = T - \{v_0, v_1, v_2\} \). Since \( d_T(v_3) > 2, v_3 \) is not a leaf in \( T' \) and since \( k \geq 5, v_3 \) is not a support vertex in \( T' \). Moreover, it is no problem to verify that \( \gamma(T') = \gamma(T) - 1 \) and \( \gamma^2(T') \geq \gamma^2(T) - 1 \). Hence

\[
\gamma^2(T) - 1 \leq \gamma^2(T') \leq \gamma(T') = \gamma(T) - 1 = \gamma^2(T) - 1.
\]

Thus, \( \gamma^2(T') = \gamma(T') \) and by the induction, \( T' \) belongs to the family \( \mathcal{I} \). Moreover, \( T \) may be obtained from \( T' \) by operation \( J_2 \) and we conclude that \( T \) also belongs to the family \( \mathcal{I} \).

Thus assume \( d_T(v_1) = d_T(v_2) = d_T(v_3) = 2 \). Without loss of generality, denote by \( D \) a minimum dominating set of \( T \) containing \( v_1 \). In this situation \( v_2, v_3 \) or \( v_4 \) belong to \( D \) to dominate \( v_3 \). If \( v_2 \) or \( v_3 \) is in \( D \), then \( D' = (D \cup \{v_2\}) - \{v_1, v_3\} \) is a 2-distance dominating set of \( T \) of cardinality smaller than \( \gamma(T) \), which is impossible. Hence \( v_4 \in D \). Observe that \( D' \), defined as above, 2-distance dominates \( v_4 \). Moreover, if \( w \) is a neighbour of \( v_4 \) and \( d_T(w, D - \{v_4\}) \leq 2 \), then \( w \) is 2-distance dominated by \( D' \) and again \( \gamma^2(T') < \gamma(T) \). Thus \( v_4 \) has a neighbour, say \( u \), such that \( d_T(u, D - \{v_4\}) \geq 3 \). Since \( T \) is a tree and each neighbour of \( u \) is dominated by \( D \), we conclude that \( u \) is a leaf and for this reason \( v_4 \) is a support vertex. Denote \( T' = T - \{v_0, v_1, v_2, v_4\} \). Since \( u \) is a leaf in \( T' \), \( v_4 \) is a support vertex in \( T' \). Moreover, it is no problem to verify that \( \gamma(T') + 1 = \gamma(T) \). Further, since \( d_T(u, v_0) = 5, \gamma^2(T') + 1 = \gamma^2(T) \). Thus, \( \gamma^2(T') = \gamma(T') \) and by the induction, \( T' \) belongs to the family \( \mathcal{I} \). Moreover, \( T \) may be obtained from \( T' \) by operation \( J_3 \) and we conclude that \( T \) also belongs to the family \( \mathcal{I} \).

The following Theorem is an immediate consequence of Lemma 6 and Corollary 5.

**Theorem 7.** Let \( T \) be a tree. Then \( \gamma(T) = \gamma^2(T) \) if and only if \( T \) belongs to the family \( \mathcal{I} \).
4. Unicyclic Graphs

A unicyclic graph is a graph that contains precisely one cycle. Our next results consider graphs with cycles.

**Lemma 8.** Let $G$ be a connected graph with $\gamma(G) = \gamma^2(G)$. If $u, v$ are two leaves of $G$ adjacent to the same support vertex, then $\gamma(G+uv) = \gamma^2(G+uv)$.

**Proof.** Let $G$ be a connected graph with $\gamma(G) = \gamma^2(G)$ and let $u, v$ be two leaves of $G$ such that $d_G(u, v) = 2$ and let $w$ be the neighbour of $u$ and $v$. By our assumptions and some immediate properties of the domination number of a graph,

$$\gamma^2(G+uv) \leq \gamma(G+uv) \leq \gamma(G) = \gamma^2(G).$$

Hence it suffices to justify that $\gamma^2(G+uv) \geq \gamma(G+uv)$. Clearly, $N_{G+uv}^2[x] = N_G^2[x]$ for each $x \in V(G)$. Thus, every minimum 2-distance dominating set of $G+uv$ is also a minimum 2-distance dominating set of $G$. Therefore, $\gamma^2(G+uv) \geq \gamma^2(G)$ and hence $\gamma(G+uv) = \gamma^2(G+uv)$.

By Theorem 7 and recursively using Lemma 8 we may obtain graphs $G$ with $\gamma(G) = \gamma^2(G)$ and containing any number of induced cycles $C_3$.

Now we characterize all connected unicyclic graphs $G$ with $\gamma(G) = \gamma^2(G)$. To this aim we introduce some additional notations. Let $T$ be a tree belonging to the family $\mathcal{T}$. We call $v \in V(T)$ an active vertex, if $v$ is a leaf adjacent to a strong support vertex or $v \in V(T) - (S(T) \cup \Omega(T))$. Further, let $\mathcal{C}_6^+$ be the family of all unicyclic graphs that may be obtained from a tree $T$ belonging to the family $\mathcal{T}$ and the cycle $C_6$ by identifying one vertex of $C_6$ with a support vertex of $T$. In addition, let $C_6$ belong to $\mathcal{C}_6^+$.

Define $\mathcal{C}$ to be the family of all unicyclic graphs that belong to $\mathcal{C}_6^+$ or may be obtained from a tree $T$ belonging to the family $\mathcal{T}$ by adding an edge between two active vertices of $T$.

The following two lemmas prove that $\gamma(G) = \gamma^2(G)$ for every graph $G$ belonging to the family $\mathcal{C}$.

**Lemma 9.** Each graph belonging to the family $\mathcal{C}_6^+$ has equal domination and 2-distance domination numbers.

**Proof.** Let $G \in \mathcal{C}_6^+$. Obviously $\gamma(C_6) = \gamma^2(C_6)$. Thus let $G$ be obtained from a tree $T$ belonging to the family $\mathcal{T}$ and the cycle $C_6 = (v_1, \ldots, v_6, v_1)$ by identifying the vertex $v_1$ with a support vertex of $T$. 
Since $G$ is unic和平 and connected, $G - v_5v_6$ is a tree. It is no problem to observe, that $G - v_5v_6$ may be obtained from $T$ by adding to $T$ first the path $P_4 = (v_2, v_3, v_4, v_5)$ and the edge $v_1v_2$, and then $v_6$ and the edge $v_1v_6$. Since $T \in \mathcal{T}$ and $G - v_5v_6$ may be obtained from $T$ by operations $\mathcal{T}_3$ and $\mathcal{T}_1$, we conclude that $G - v_5v_6 \in \mathcal{T}$. Thus by Lemma 4, $\gamma^2(G - v_5v_6) = |S(G - v_5v_6)|$
and by Lemma 5, $\gamma(G - v_5v_6) = \gamma^2(G - v_5v_6)$.

Let $D$ be a $\gamma^2(G)$-set. Since $G$ is obtained from $T$ and $C_6$ by identifying $v_1$ with a support vertex of $T$ and $\gamma^2(T) = |S(T)|$, $|D| \geq |S(T)|$. Denote by $x$ a leaf adjacent to $v_1$ in $G$. Then there exists a vertex $y$ such that $y \in N_G^2(x) \cap D$. In any choice of $y$, at least one vertex belonging to $\{v_1, \ldots, v_6\} - \{y\}$ belongs also to $D$ (because $D$ is 2-distance dominating). Thus $|D| \geq |S(T)| + 1$. On the other hand, $S(G) \cup \{v_4\}$ is a 2-distance dominating set of $G$ of cardinality $|S(G)| + 1$. Thus

$$|S(G)| + 1 = \gamma^2(G) \leq \gamma(G) \leq \gamma(G - v_5v_6) = \gamma^2(G - v_5v_6) = |S(G - v_5v_6)|.$$  

(1)

Since $|S(G)| = |S(G - v_5v_6)| - 1$, we have equalities throughout the inequality chain (1). In particular, $\gamma^2(G) = \gamma(G)$.

**Lemma 10.** If $G$ is a graph obtained from a tree $T$ belonging to the family $\mathcal{T}$ by adding an edge between two active vertices of $T$, then $\gamma(G) = \gamma^2(G)$.

**Proof.** Let $T$ be a tree belonging to the family $\mathcal{T}$. Denote by $u$ and $v$ two active vertices of $T$ and let $D$ be a $\gamma^2(G)$-set, where $G = T + uv$. If $u$ and $v$ are leaves adjacent to the same support vertex, then the result follows from Lemma 8.

Thus assume $u$ and $v$ are adjacent to different support vertices of $T$ or at most one of $u$ and $v$ is a leaf. In both cases, $S(T) = S(G)$ and similarly like in $T$, the distance between any two leaves adjacent to different support vertices in $G$ is greater than or equal to 5. For this reason, if $u$ and $v$
are two leaves adjacent to different support vertices in $G$, then $u$ and $v$
 cannot be $2$-distance dominated by the same element of $D$. This implies
that $\gamma^2(G) \geq |S(G)|$. Hence

$$|S(G)| \leq \gamma^2(G) \leq \gamma(G) \leq \gamma(T) = \gamma^2(T) = |S(T)| = |S(G)|.$$ 

Therefore $\gamma(G) = \gamma^2(G)$. 

For a cycle $C_n$ on $n \geq 3$ vertices it is no problem to see that $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$ and $\gamma^2(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

**Lemma 11.** If $G$ is a connected unicyclic graph with $\gamma(G) = \gamma^2(G)$, then $G$ belongs to the family $\mathcal{C}$.

**Proof.** Let $G$ be a unicyclic graph, where $C_k = (v_1, \ldots, v_k, v_1)$ is the unique
cycle of $G$. If $d_G(v_i) > 2$ for some $v_i \in V(C_k)$, then let $T(v_i)$ be the tree
attached to the vertex $v_i$ and let $v_i$ be the root of $T(v_i)$. Let $D$ be a minimum
dominating set of $G$ containing all support vertices of $G$.

By Proposition 1, at most $\left\lfloor \frac{k}{3} \right\rfloor$ vertices of $C_k$ belong to $D$ and the
distance between any two elements of $D$ is at least $3$. Thus there exists an
edge, without loss of generality say $v_2v_3$ (where $v_2, v_3 \in V(C_k)$), such that
$v_2 \notin D$ and $v_3 \notin D$. Note that neither $v_2$ nor $v_3$ is a support vertex. Since $G$
is unicyclic and connected, $G - v_2v_3$ is a tree. Moreover, by our assumptions
and some immediate properties of the domination number of a graph,

$$\gamma(G) = \gamma^2(G) \leq \gamma^2(G - v_2v_3) \leq \gamma(G - v_2v_3). \quad (2)$$

However, since $v_2, v_3 \notin D$, $D$ is also a dominating set in $G - v_2v_3$. Therefore,$\gamma(G) = \gamma(G - v_2v_3)$ and thus we have equalities throughout the inequality
chain (2). In particular, $\gamma^2(G - v_2v_3) = \gamma(G - v_2v_3)$ and since $G - v_2v_3$
is a tree, Theorem 7 implies that $G - v_2v_3$ belongs to the family $\mathcal{T}$. By
Obsevation 3, each vertex of $G - v_2v_3$ is a support vertex or is a neighbour
of exactly one support vertex. Of course $v_2, v_3 \notin S(G - v_2v_3)$. Hence
denote by $s_2$ and $s_3$ the support vertices adjacent in $G - v_2v_3$ to $v_2$ and $v_3$,$\,$respectively. Observe that $s_2$ and $s_3$ may not be support vertices in $G$.

If $s_2 = s_3$, then $v_1 = s_2$. If $v_1$ is a support vertex in $G$, then $G$
may be obtained from the tree $G - v_2v_3$ by adding an edge between two
active vertices adjacent to the same support vertex and thus $G \in \mathcal{C}$. If
$v_1 \notin S(G)$, then at least one of $v_2, v_3$ is of degree $2$ in $G$. Assume first
d$G(v_2) = d_G(v_3) = 2$. Then $v_2$ and $v_3$ are leaves in $G - v_2v_3$ and for
this reason $G$ again may be obtained from the tree $G - v_2v_3$ by adding an edge between two active vertices. Thus assume, without loss of generality, $d_G(v_2) = 2$ and $d_G(v_3) \geq 3$. Observe that since $v_1 \notin S(G)$, every element of $V(G) - \{v_1, v_2\}$ is within distance 2 from a vertex belonging to $D - \{v_1\}$. Thus, $D - \{v_1\}$ 2-distance dominates $V(G) - \{v_1, v_2\}$. Denote by $x$ an element of $D \cap V(T(v_3))$, which is at distance 3 from $v_1$ and let $(x, y, v_3, v_1)$ be the shortest path from $x$ to $v_1$. Define $D' = (D - \{x, v_1\}) \cup \{y\}$. Now every element of $V(G)$ is within distance 2 from an element of $D'$, so $D'$ is a 2-distance dominating set of $G$ smaller than $\gamma(G)$, which contradicts that $\gamma(G) = \gamma^2(G)$.

In what follows we assume $s_2 \neq s_3$ and we consider three cases.

1. If $s_2 \in S(G)$ and $s_3 \in S(G)$, then $v_2$ and $v_3$ are both active vertices in $G - v_2v_3$. Therefore $G$ may be obtained from the tree $G - v_2v_3$ by adding the edge $v_2v_3$ and thus $G$ belongs to the family $\mathcal{C}$.

2. Without loss of generality, assume that $s_2 \notin S(G)$ and $s_3 \in S(G)$. Thus, $v_2$ is the unique leaf adjacent to $s_2$ in $G - v_2v_3$. Therefore $d_G(v_2) = 2$ and $s_2 = v_1$. Observe, that since $v_1 \notin S(G)$, each element of $V(G) - \{v_1\}$ is within distance 2 from an element of $D - \{v_1\}$. Thus, $D - \{v_1\}$ 2-distance dominates $V(G) - \{v_1\}$.

   If $d_G(v_1) \geq 3$, then since $v_1$ is not a support vertex in $G$, $D \cap V(T(v_1)) \neq \emptyset$. Denote by $x$ an element of $D \cap V(T(v_1))$, which is at distance 3 from $v_1$ and let $(x, y, z, v_1)$ be the shortest path from $x$ to $v_1$. Define $D' = (D - \{x, v_1\}) \cup \{y\}$. It is no problem to see that $D'$ is a 2-distance dominating set of $G$, which contradicts that $\gamma(G) = \gamma^2(G)$. We conclude that $d_G(v_1) = 2$.

   If $s_3 \neq v_3$, then $d_G(v_3) \geq 3$. Define $D' = (D - \{s_3\}) \cup \{v_3\}$. Then, since $d_G(v_1, v_3) = 2$, $D' - \{v_1\}$ is a 2-distance dominating set of $G$, contradicting that $\gamma(G) = \gamma^2(G)$. We conclude that $s_3 = v_4$ and since $v_3$ is a support vertex, $d_G(v_4) \geq 3$ and $v_1 \neq v_4$. Moreover, $v_5, v_6 \notin D$ and for this reason $v_5, v_6 \notin S(G)$. Denote by $v_0$ a vertex belonging to $D$ and at distance 2 from $v_k$. If $v_0 \neq v_k$, then $(D - \{v_1, v_4\}) \cup \{v_3\}$ is a 2-distance dominating set of $G$ of smaller cardinality than $\gamma(G)$, a contradiction. Therefore, $v_0 = v_4$ and since $d_G(v_4, v_k) = 2$ we obtain $v_k = v_0$.

We have already proven, that under our conditions $d_G(v_1) = d_G(v_2) = 2$ and $v_4$ is a support vertex. Suppose $d_G(v_6) \geq 3$. Then since $v_6$ is not a support vertex in $G$, $D \cap V(T(v_6)) \neq \emptyset$. Denote by $x$ an element of $D \cap V(T(v_6))$, which is at distance 3 from $v_1$ and let $(x, y, v_6, v_1)$ be the shortest path from $x$ to $v_1$. Define $D' = (D - \{x, v_1\}) \cup \{y\}$. Now $D'$ is
a 2-distance dominating set of $G$, which contradicts that $\gamma(G) = \gamma^2(G)$. Therefore $d_G(v_6) = 2$.

Suppose $d_G(v_5) \geq 3$. Then since $v_5$ is not a support vertex in $G$, $D \cap V(T(v_5)) \neq \emptyset$. Denote by $x$ an element of $D \cap V(T(v_5))$, which is at distance 3 from $v_4$ and let $(x, y, v_5, v_4)$ be the shortest path from $x$ to $v_4$. Define $D' = (D - \{x, v_1, v_4\}) \cup \{y, v_3\}$. Now $D'$ is a 2-distance dominating set of $G$, which contradicts that $\gamma(G) = \gamma^2(G)$. Therefore $d_G(v_5) = 2$.

Similarly we prove that $d_G(v_3) = 2$.

Therefore, $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_5) = d_G(v_6) = 2$ and $v_4$ is a support vertex. Hence $G$ may be obtained from a tree $T$ and the cycle $C_6$ by identifying one vertex of $C_6$ with a support vertex of $T$. Clearly, $D - \{v_1\}$ is a dominating set of $T$, so

$$\gamma^2(T) \leq \gamma(T) \leq \gamma(G) - 1 = \gamma^2(G) - 1. \quad (3)$$

On the other hand, any 2-distance dominating set of $T$ may be extended to a dominating set of $G$ by adding to it $v_1$. Thus $\gamma^2(G) \leq \gamma^2(T) + 1$ and we have equalities through the inequality chain (3). In particular, $\gamma^2(T) = \gamma(T)$.

By Theorem 7, $T$ belongs to the family $\mathcal{T}$. Hence $G$ may be obtained from $T \in \mathcal{T}$ and the cycle $C_6$ by identifying one vertex of $C_6$ with a support vertex of $T$. Thus $G \in \mathcal{C}_6^+$.

3. If $s_2 \notin S(G)$ and $s_3 \notin S(G)$, then $d_G(v_2) = 2$ and $d_G(v_3) = 2$. Moreover, $v_1 = s_2$ and $v_4 = s_3$. Since $v_1$ is not a support vertex, each element of $V(G) - \{v_1\}$ is within distance 2 from an element of $D - \{v_1\}$. Hence, $D - \{v_1\}$ 2-distance dominates $V(G) - \{v_1\}$. By the same reasoning, $D - \{v_4\}$ 2-distance dominates $V(G) - \{v_4\}$. Similarly as in previous case, we deduce that $d_G(v_1) = d_G(v_4) = 2$. Since $v_1 \neq v_4$, the unique cycle contains at least 6 vertices, $v_5, v_6 \notin D$ and $v_5, v_6 \notin S(G)$.

If $d_G(v_5) \geq 3$, then since $v_5$ is not a support vertex, $D \cap V(T(v_5)) \neq \emptyset$. Denote by $x$ an element of $D \cap V(T(v_5))$, which is at distance 3 from $v_4$ and let $(x, y, v_5, v_4)$ be the shortest path from $x$ to $v_4$. Define $D' = (D - \{x, v_4\}) \cup \{y\}$. Now $D'$ is a 2-distance dominating set of $G$, which contradicts that $\gamma(G) = \gamma^2(G)$. Therefore $d_G(v_5) = 2$.

Since $D$ is dominating, $v_6$ has a neighbour in $D$. If there exists $x \in N_G(v_6) \cap D$ such that $x \neq v_1$, then $(D - \{v_1, v_4\}) \cup \{v_3\}$ is a 2-distance dominating set of $G$, which contradicts that $\gamma(G) = \gamma^2(G)$. Thus we conclude that $\{v_1\} = N_G(v_6) \cap D$. Therefore the unique cycle of $G$ contains exactly 6 vertices. By similar reasoning as for $v_5$, we obtain that $d_G(v_6) = 2$. Hence
each vertex of the unique cycle is of degree 2 and $G = C_2$. Therefore $G$ belongs to the family $\mathcal{C}$.

The following results are consequences of Theorem 7 and Lemmas 9 and 11.

**Theorem 12.** Let $G$ be a connected unicyclic graph. Then $\gamma(G) = \gamma^2(G)$ if and only if $G$ belongs to the family $\mathcal{C}$.

**Theorem 13.** Let $G$ be a unicyclic graph. Then $\gamma(G) = \gamma^2(G)$ if and only if exactly one connected component of $G$ is a unicyclic graph belonging to the family $\mathcal{C}$ and each other connected component of $G$ is a tree belonging to the family $\mathcal{T}$.

**References**


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