ON THE EXISTENCE OF SOLUTIONS OF AN
INTEGRO-DIFFERENTIAL EQUATION
IN BANACH SPACES

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Dedicated to Professor Michal Kisielewicz
on the occasion of his 70th birthday

Consider the Cauchy problem

\begin{align}
  x^{(m)}(t) &= f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \\
  x(0) &= 0, x'(0) = \eta_1, \ldots, x^{(m-1)}(0) = \eta_{m-1}
\end{align}

in a Banach space $E$, where $m \geq 1$ is a natural number. We assume that $D = [0, a]$, $B = \{x \in E : \|x\| \leq b\}$ and $f : D \times B \to E$, $g : D^2 \times B \to E$ are bounded continuous functions. Let

\[ m_1 = \sup \{\|f(t, x)\| : t \in D, x \in B\} \]

\[ m_2 = \sup \{\|g(t, s, x)\| : t, s \in D, x \in B\}. \]

We choose a positive number $d$ such that $d \leq a$ and

\[ \sum_{j=1}^{m-1} \frac{\|\eta_j\|}{j!} + m_1 \frac{d^m}{m!} + m_2 \frac{d^{m+1}}{m!} \leq b. \]

Let $J = [0, d]$. Denote by $C = C(J, E)$ the Banach space of continuous functions $z : J \to E$ with the usual norm $\|z\|_C = \max_{t \in J} \|z(t)\|$. Let $\tilde{B} = \{x \in C : \|x\|_C \leq b\}$. For $t \in J$ and $x \in \tilde{B}$ put

\[ \tilde{g}(t, x) = \int_0^t g(t, s, x(s))ds. \]
Fix $\tau \in J$ and $x \in \tilde{B}$. As the set $J \times x(J)$ is compact, from the continuity of $g$ it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that
\[
\|g(t, s, x(s)) - g(\tau, s, x(s))\| < \varepsilon \quad \text{for} \quad t, s \in J \quad \text{with} \quad |t - \tau| < \delta.
\]
In view of the inequality
\[
\|\tilde{g}(t, x) - \tilde{g}(\tau, x)\| \leq m_2|t - \tau| + \int_0^\tau \|g(t, s, x(s)) - g(\tau, s, x(s))\| ds,
\]
this implies the continuity of the function $t \to \tilde{g}(t, x)$. On the other hand, the Lebesgue dominated convergence theorem proves that for each fixed $t \in J$ the function $x \to \tilde{g}(t, x)$ is continuous on $\tilde{B}$. Moreover,
\[
\|\tilde{g}(t, x)\| \leq m_2t \quad \text{for} \quad t \in J \quad \text{and} \quad x \in \tilde{B}.
\]
Let $\alpha$ be the Kuratowski measure of noncompactness in $E$ (cf. [1]).

The main result of the paper is the following

**Theorem.** Let $w : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous nondecreasing function such that $w(0) = 0$, $w(r) > 0$ for $r > 0$ and
\[
\int_{0^+}^\infty \frac{dr}{\sqrt[2]{r^{m-1}w(r)}} = \infty.
\]
If
\[(4) \quad \alpha(f(t, X)) \leq w(\alpha(X)) \quad \text{for} \quad t \in J \quad \text{and} \quad X \subset B,
\]
and the set $g(D^2 \times B)$ is relatively compact in $E$, then there exists at least one solution of (1)–(2) defined on $J$.

**Proof.** The problem (1)–(2) is equivalent to the integral equation
\[
x(t) = p(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}[f(s, x(s)) + \tilde{g}(s, x)]ds \quad (t \in J),
\]
where $p(t) = \sum_{j=1}^{m-1} \eta_j \frac{t^j}{j!}$. We define the mapping $F$ by
\[
F(x)(t) = p(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}[f(s, x(s)) + \tilde{g}(s, x)]ds \quad (t \in J, x \in \tilde{B}).
\]
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Owing to (3), it is known (cf. [5]) that $F$ is a continuous mapping $\hat{B} \mapsto \hat{B}$ and the set $F(\hat{B})$ is equicontinuous. By the Mazur lemma the set $W = \bigcup_{0 \leq \lambda \leq d} \text{conv}(D^2 \times B)$ is relatively compact. Since $\{(t-s)^{m-1}\bar{g}(s, x) : x \in \hat{B}\} \subseteq (t-s)^{m-1}W$, we have $\alpha(\{(t-s)^{m-1}\bar{g}(s, x) : x \in \hat{B}\}) \leq (t-s)^{m-1}\alpha(W) = 0$. Therefore, by the Heinz lemma [2]

\[
\alpha \left( \left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}\bar{g}(s, x)ds : x \in \hat{B} \right\} \right) \\
\leq \frac{2}{(m-1)!} \int_0^t \alpha \left( \left\{ (t-s)^{m-1}\bar{g}(s, x) : x \in \hat{B} \right\} \right) \, ds = 0.
\]

(5)

For any positive integer $n$ put

\[
v_n(t) = \begin{cases} 
p(t) & \text{if } 0 \leq t \leq \frac{d}{n} \\
p(t) + \frac{1}{(m-1)!} \int_0^{t-d/n} (t-s)^{m-1}[f(s, v_n(s)) + \bar{g}(s, v_n)]ds & \text{if } \frac{d}{n} \leq t \leq d.
\end{cases}
\]

Then, by (3), $v_n \in \hat{B}$ and

(6) \[\lim_{n \to \infty} \| v_n - F(v_n) \|_C = 0.\]

Put $V = \{v_n : n \in N\}$ and $Z(t) = \{x(t) : x \in Z\}$ for $t \in J$ and $Z \subseteq C$. As $V \subseteq \{v_n - F(v_n) : n \in N\} + F(V)$ and $V \subseteq \hat{B}$, from (6) it follows that the set $V$ is equicontinuous and the function $t \mapsto v(t) = \alpha(V(t))$ is continuous on $J$. Applying now the Heinz lemma and (5), we get

\[
\alpha(F(V)(t)) = \\
= \alpha \left( \left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}[f(s, v_n(s)) + \bar{g}(s, v_n)]ds : n \in N \right\} \right) \\
\leq \alpha \left( \left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}f(s, v_n(s))ds : n \in N \right\} \right) \\
+ \alpha \left( \left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}\bar{g}(s, x)ds : x \in \hat{B} \right\} \right)
\]
\[
\alpha \left( \left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f(s, v_n(s)) ds : n \in N \right\} \right) = \alpha \left( \left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f(s, v_n(s)) ds : n \in N \right\} \right)
\]

\[
\leq \frac{2}{(m-1)!} \int_0^t \alpha \left( \left\{ (t-s)^{m-1} f(s, v_n(s)) : n \in N \right\} \right) ds
\]

\[
\leq \frac{2}{(m-1)!} \int_0^t (t-s)^{m-1} \alpha(f(s, V(s))) ds
\]

\[
\leq \frac{2}{(m-1)!} \int_0^t (t-s)^{m-1} w(\alpha(V(s))) ds.
\]

On the other hand, from (6) and the inclusion

\[
V(t) \subset \{v_n(t) - F(v_n)(t) : n \in N\} + F(V(t))
\]

it follows that \( v(t) \leq \alpha(F(V(t))) \). Hence

\[
v(t) \leq \frac{2}{(m-1)!} \int_0^t (t-s)^{m-1} w(v(s)) ds \text{ for } t \in J.
\]

Putting \( h(t) = \frac{2}{(m-1)!} \int_0^t (t-s)^{m-1} w(v(s)) ds \), we see that \( h \in C^m \), \( v(t) \leq h(t) \), \( h^{(j)}(t) \geq 0 \) for \( j = 0, 1, \ldots, m \), \( h^{(j)}(0) = 0 \) for \( j = 0, 1, \ldots, m - 1 \) and \( h^{(m)}(t) = 2w(v(t)) \leq 2w(h(t)) \) for \( t \in J \). By Theorem 1 of [6], from this we deduce that \( h(t) = 0 \) for \( t \in J \). Thus \( \alpha(V(t)) = 0 \) for \( t \in J \). Therefore for each \( t \in J \) the set \( V(t) \) is relatively compact in \( E \), and by Ascoli’s theorem the set \( V \) is relatively compact in \( C \). Hence we can find a subsequence \((v_{n_k})_k\) of \((v_n)\) which converges in \( C \) to a limit \( u \). As \( F \) is continuous, from (6) we conclude that \( u = F(u) \), so that \( u \) is a solution of (1)–(2).

**Remark.** It is known (cf. [7], Theorem 4) that under the assumptions of the Theorem the set of all solutions of (1)–(2) defined on \( J \) is a compact \( R_\delta \) set in \( C(J, E) \).

**References**


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