Abstract

In this paper, we prove that the topological dual of the Banach space of bounded measurable functions with values in the space of nuclear operators, furnished with the natural topology, is isometrically isomorphic to the space of finitely additive linear operator-valued measures having bounded variation in a Banach space containing the space of bounded linear operators. This is then applied to a stochastic structural control problem. An optimal operator-valued measure, considered as the structural control, is to be chosen so as to minimize fluctuation (volatility). Both existence of optimal policy and necessary conditions of optimality are presented including a conceptual algorithm.

Keywords: representation theory, topological dual, finitely additive operator-valued measures, polish space, Hilbert space, stochastic systems, structural control, uncertainty abatement.


1. Introduction

The subject of representation of continuous linear functionals on topological spaces is one of the central topic in functional analysis. The best example, widely known, is the celebrated Riesz representation theorems and their broad extensions. These results are found in Dunford and Schwartz [2]
Theorem IV.5.1, Corollary IV.5.3, Theorem IV.6.2 and Theorem IV.6.3. In physical sciences and engineering there are many problems found in the study of optimal control and optimization that require continuous linear functionals on the space of operator-valued functions. This was the main motivation for this study and we believe that it may have some significant bearing on functional analysis on the space of operator-valued functions. This assertion is very well illustrated by an example from infinite dimensional stochastic systems on Hilbert space.

The rest of the paper is organized as follows. In Section 2, some basic notations are introduced. In Section 3, the main representation theorem is proved. In Section 4, this result is used to study a class of stochastic systems on Hilbert space furnished with structural controls represented by operator-valued measures which are subject to some natural constraints. These controls can be chosen optimally to improve the performance of the system by way of reducing volatility and hence uncertainty introduced by the presence of continuous as well as jump martingales. The paper is concluded after Section 5 where computational algorithms for some special cases are discussed.

2. Some notations

Let \( \{X,Y\} \) denote a pair of real separable Hilbert spaces with a corresponding pair of complete orthonormal basis denoted by \( \{x_i, y_i\}, i \in \mathbb{N} \). Let \( \mathcal{L}(Y,X) \) denote the space of linear operators (not necessarily bounded) from \( Y \) to \( X \) and \( \mathcal{L}(Y,X) \subseteq \mathcal{L}(Y,X) \) the space of bounded linear operators from \( Y \) to \( X \). Furnished with the uniform operator topology, \( \mathcal{L}(Y,X) \) is a Banach space. Let \( \mathcal{L}_1(X,Y) \subseteq \mathcal{L}(X,Y) \) denote the space of nuclear operators from \( X \) to \( Y \) having the representation

\[
L \equiv \sum_{i,j=1}^{\infty} \zeta_{i,j} \ y_j \otimes x_i
\]

with the scalars \( \{\zeta_{i,j}\} \) satisfying

\[
\sum_{i,j} |\zeta_{i,j}| < \infty.
\]

This is furnished with the nuclear norm (trace norm) topology given by

\[
\|L\|_1 \equiv \sum_{i,j=1}^{\infty} |(Lx_i, y_j)|,
\]
whenever it is defined. It is easy to verify that this norm is independent of the choice of the orthonormal basis. With respect to this norm topology, $L_1(X,Y)$ is a Banach space. Since the pair of Hilbert spaces $\{X,Y\}$ are assumed to be separable, $L_1(X,Y)$ is a separable Banach space. Let us introduce the following linear subspace $L_0(Y,X)$ of the space $L(Y,X)$ as follows:

$$L_0(Y,X) \equiv \{ B \in L(Y,X) : \| B \|_0 < \infty \}$$

where

$$\| B \|_0 \equiv \sup_{\{k,t\} \in N} |(By_k, x_t)_X|.$$ 

It is easy to verify that this defines a norm on $L_0(Y,X)$ and that this norm is generated by an increasing family of seminorms $\{\rho_n\}$ given by $\rho_n(B) \equiv \sup\{|(By_j, x_i)_X|, 1 \leq i,j \leq n\}, n \in N$. Furnished with the topology induced by this norm, $L_0(Y,X)$ is a normed linear space containing the Banach space of bounded linear operators $L(Y,X)$ with continuous embedding

$$L(Y,X) \hookrightarrow L_0(Y,X).$$

For economy of notations, we continue to use $L_0(Y,X)$ to denote its completion with respect to the norm topology defined above turning it into a Banach space. Now we introduce the following function spaces. Let $I$ be a finite interval of the real line with $\Sigma \equiv \sigma(I)$ denoting the sigma algebra of subsets of the set $I$. Let $S_\infty(I, L_1(X,Y))$ denote the class of all $\Sigma$ measurable simple functions defined on $I$ and taking values from the Banach space $L_1(X,Y)$. Clearly this is a vector space. We furnish this with the norm topology

$$\| L \|_\infty \equiv \sup\{ \| L(t) \|_{L_1(X,Y)} \equiv \| L(t) \|_1, t \in I \}.$$ 

Let $B_\infty(I, L_1(X,Y))$ denote the closure of $S_\infty(I, L_1(X,Y))$ in the norm topology defined above. Clearly this is a Banach space and $S_\infty(I, L_1(X,Y))$ is dense in $B_\infty(I, L_1(X,Y))$.

Let $M_{ba}(\Sigma)$ denote the class of (real) bounded finitely additive signed measures on $I$. It is well known [1, 2] that, furnished with the total variation norm, this is a Banach space. We are interested in the class of finitely additive $L_0(Y,X)$ valued vector measures defined on $\Sigma \equiv \sigma(I)$. We denote this class by $M_{ba}(\Sigma, L_0(Y,X))$, and furnish this with the norm topology
given by
\[ \| B \|_{M_{ba}(\Sigma, L_0(Y,X))} \equiv \sup_{\pi} \sum_{\sigma \in \pi} \| (B(\sigma)) \|_0, \]

where the supremum outside the summation sign is taken over all finite \( \Sigma \) measurable disjoint partitions \( \pi \) of the interval \( I \). With respect to this topology, \( M_{ba}(\Sigma, L_0(Y,X)) \) is also a Banach space. Readers, interested in vector measures, may like to see the celebrated books of Diestel and Uhl Jr [1] and Dunford and Schwartz [2].

3. Representation theorem

Our main result of this section is a representation theorem characterizing the topological dual of \( B_\infty(I, L_1(X,Y)) \), in other words, continuous linear functionals on \( B_\infty(I, L_1(X,Y)) \). This is applied in Section 4 where we study control problems for stochastic systems on Hilbert spaces. Operator-valued measures are used as structural controls to combat fluctuation.

**Theorem 3.1.** The topological dual of the Banach space \( B_\infty(I, L_1(X,Y)) \) is isometrically isomorphic to the space \( M_{ba}(\Sigma, L_0(Y,X)) \). We may express this by writing
\[ (B_\infty(I, L_1(X,Y)))^* \cong M_{ba}(\Sigma, L_0(Y,X)), \]

where \( (B_\infty(I, L_1(X,Y)))^* \) is the topological dual of \( B_\infty(I, L_1(X,Y)) \).

**Proof.** Without loss of generality, we may assume throughout the proof that the pair of complete orthonormal basis \( \{x_i, y_i\} \) for the pair of (separable) Hilbert spaces \( \{X, Y\} \), respectively, is fixed. Let \( \ell \in (B_\infty(I, L_1(X,Y)))^* \). For any \( L \in S_\infty(I, L_1(X,Y)) \), there exist \( n \in N \), a disjoint \( \Sigma \)-measurable partition \( \pi_n \equiv \{\Delta_i\}^n_{i=1} \) of \( I \) and \( \{L_i\}^n_{i=1} \in L_1(X,Y) \) such that \( L(t) \equiv \sum_{i=1}^n \chi_{\Delta_i}(t)L_i \) where \( \chi_\sigma \) is the characteristic function of the set \( \sigma \in \Sigma \). The value of \( \ell \) at \( L \) is given by
\[ \ell(L) = \ell\left( \sum_{i=1}^n \chi_{\Delta_i}L_i \right) = \sum_{i=1}^n \ell(\chi_{\Delta_i}L_i). \]

Clearly, \( \chi_{\Delta_i}L_i \in B_\infty(I, L_1(X,Y)) \) with norm \( \| \chi_{\Delta_i}L_i \|_\infty = \| L_i \|_{L_1(X,Y)} \). Define the linear functional \( \ell_{\Delta_i} \) on \( L_1(X,Y) \) by \( \ell_{\Delta_i}(G) \equiv \ell(\chi_{\Delta_i}G), G \in L_1(X,Y) \). Since \( \ell \) is a continuous linear functional on \( B_\infty(I, L_1(X,Y)) \),
it is clear that $\ell_{i}$ is a continuous linear functional on $L_1(X,Y)$. With respect to the fixed orthonormal basis $\{x_i\}$ of $X$ and $\{y_i\}$ of $Y$, $i \in \mathbb{N}$, as introduced above, every continuous linear functional $\eta$ on $L_1(X,Y)$ has the representation
\[
\eta(G) = Tr(CG) = \sum_{k,j=1}^{\infty} \zeta_{k,j} \; (Cy_j, x_k)
\]
for some $C \in L_0(Y,X)$ where $G = \sum_{i,j=1}^{\infty} \zeta_{i,j} \; y_j \otimes x_i$ with the trace (nuclear) norm given by
\[
\| G \|_1 = \sum_{i,j=1}^{\infty} |\zeta_{i,j}| < \infty.
\]
The operator $C \in L_0(Y,X)$ is uniquely determined by $\eta$ alone. Hence $\ell_{\Delta_i}$ must have the form
\[
\ell_{\Delta_i}(L_i) = \sum_{i,j=1}^{\infty} \zeta_{i,j}^{(i)} \; (C_i y_j, x_k)
\]
for some $C_i \in L_0(Y,X)$ uniquely determined by $\ell_{\Delta_i}$ alone with $L_i$ given by
\[
L_i = \sum_{i,j=1}^{\infty} \zeta_{i,j}^{(i)} \; y_j \otimes x_k.
\]
Thus there exists a set function $B : \Sigma \rightarrow L_0(Y,X)$ such that for each $i \in \{1, 2, \cdots, n\}$, $B(\Delta_i) = C_i$, and for the empty set $\emptyset$, $B(\emptyset) = 0$, is the zero operator. Since $L_i \in L_1(X,Y)$ for each $i = 1, 2, \cdots, n$, it is clear that the nuclear norm $\| L_i \|_1 = \sum_{k,j=1}^{\infty} |\zeta_{k,j}^{(i)}| < \infty$ for every $i = 1, 2, \cdots, n$. Clearly,
\[
(1) \quad \ell(L) = \sum_{i=1}^{n} \ell(\chi_{\Delta_i} L_i) = \sum_{i=1}^{n} \ell_{\Delta_i}(L_i) = \sum_{i=1}^{n} \left( \sum_{k,j=1}^{\infty} \zeta_{k,j}^{(i)} \; (C_i y_j, x_k) \right)
\]
\[
= \sum_{i=1}^{n} \left( \sum_{k,j=1}^{\infty} \zeta_{k,j}^{(i)} \; (B(\Delta_i) y_j, x_k) \right).
\]
By hypothesis $\ell \in (B_\infty(I, L_1(X,Y)))^*$, and so there exists a constant $c > 0$ such that
\[
(2) \quad |\ell(L)| = \left| \sum_{i=1}^{n} \left( \sum_{k,j=1}^{\infty} \zeta_{k,j}^{(i)} \; (B(\Delta_i) y_j, x_k) \right) \right| \leq c \| L \|_\infty,
\]
where \( \| L \|_\infty \equiv \sup \{ \| L(t) \|_{\mathcal{L}_1(X,Y)}, t \in I \} = \sup_{1 \leq i \leq n} \| L_i \|_1 < \infty \). Since this inequality holds for arbitrary \( L \in S_\infty(I, \mathcal{L}_1(X,Y)) \) and hence arbitrary 
\( \{ \{ \zeta^{(i)}_{k,j} \}_{k,j=1}^{\infty}, 1 \leq i \leq n \} \) satisfying

\[
\sup \left\{ \sum_{k,j=1}^{\infty} |\zeta^{(i)}_{k,j}|, 1 \leq i \leq n \right\} < \infty,
\]

we may conclude that

\[
\sum_{i=1}^{n} \sup \left\{ \| (B(\Delta_i)) y_j, x_k \|, j, k \in N \right\} \equiv \sum_{i=1}^{n} \| B(\Delta_i) \|_0 \leq c,
\]

for every finite \( n \). This is true for every \( L \in S_\infty(I, \mathcal{L}_1(X,Y)) \) and hence for any finite disjoint \( \Sigma \)-measurable partition \( \pi \) of the interval \( I \). Hence, the inequality (3) is valid for every \( n \in N \), and so we may conclude that the set function \( B : \Sigma \rightarrow \mathcal{L}_0(Y,X) \) is finitely additive on \( \Sigma \) and has a bounded variation. Thus \( B \in M_{ba}(\Sigma, \mathcal{L}_0(Y,X)) \), and it is uniquely determined by \( \ell \) alone. Using this measure and any finite disjoint \( \Sigma \)-measurable partition \( \pi = \{ \sigma_i, 1 \leq i \leq n, n \in N \} \) of the interval \( I \), one can rewrite the expression (1) in terms of the trace as follows

\[
\ell(L) = \sum_{i=1}^{n} Tr(L_i B(\sigma_i)) = \int I Tr(L(t)B(dt)).
\]

Clearly, it follows from (2) and (3) that

\[
|\ell|_* \equiv \sup \{ \| \ell(L) \|, \| L \|_\infty \leq 1 \} \leq \| B \|_{M_{ba}(\Sigma, \mathcal{L}_0(Y,X))} \leq c.
\]

Since \( S_\infty(I, \mathcal{L}_1(X,Y)) \) is dense in \( B_\infty(I, \mathcal{L}_1(X,Y)) \), it follows from (4) and (5) that for every \( L \in B_\infty(I, \mathcal{L}_1(X,Y)) \)

\[
\ell(L) = \int I Tr(L(t)B(dt))
\]

and that \( |\ell(L)| \leq c \| L \|_\infty \). Thus we have proved that, for every \( \ell \in (B_\infty(I, \mathcal{L}_1(X,Y))^*, \) there exists a \( B \in M_{ba}(\Sigma, \mathcal{L}_0(Y,X)) \) such that

\[
\ell(L) = \int I Tr(L(t)B(dt)).
\]
From this one can also verify that $\ell$ determines $B$ uniquely. Now, choosing $c = \inf\{\alpha \geq 0 : |\ell(T)| \leq \alpha, \|T\|_\infty = 1\}$, it follows from (5) that $|\ell|_* = \|B\|_{\mathcal{L}_0(Y,X)}$ and hence we conclude that the injection

$$\tag{7} \left(\mathcal{B}_\infty(I, \mathcal{L}_1(X,Y))\right)^* \hookrightarrow \mathcal{M}_{ba}(\Sigma, \mathcal{L}_0(Y,X))$$

is an isometry. Now we prove the reverse inclusion. Let $B \in \mathcal{M}_{ba}(\Sigma, \mathcal{L}_0(Y,X))$ and define the linear functional on $\mathcal{S}_\infty(I, \mathcal{L}_1(X,Y)) \subset \mathcal{B}_\infty(I, \mathcal{L}_1(X,Y))$ by

$$\tag{8} \ell_B(T) \equiv \int_I \text{Tr}(\tau(t) B(dt)), T \in \mathcal{S}_\infty(I, \mathcal{L}_1(X,Y)).$$

Since $T \in \mathcal{S}_\infty(I, \mathcal{L}_1(X,Y))$, there exists $n \in \mathbb{N}$, a disjoint partition $\pi_n = \{\sigma_i\}_{i=1}^n \in \Sigma$ of $I$ and $T_i \in \mathcal{L}_1(X,Y)$, $i = 1, 2, \cdots n$, such that $T(t) = \sum_{i=1}^n \chi_{\sigma_i}(t) T_i$ and

$$\tag{9} \ell_B(T) = \sum_{i=1}^n \text{Tr}(T_i B(\sigma_i)).$$

Hence

$$|\ell_B(T)| \leq \sum_{i=1}^n |\text{Tr}(T_i B(\sigma_i))| \leq \sum_{i=1}^n \|T_i\|_{\mathcal{L}_1(X,Y)} \|B(\sigma_i)\|_{\mathcal{L}_0(Y,X)}$$

$$\leq \left( \sup_{1 \leq i \leq n} \|T_i\|_1 \right) \left( \sum_{i=1}^n \|B(\sigma_i)\|_0 \right) \leq (\|T\|_\infty) \|B\|_{\mathcal{L}_0(Y,X)}) \cdot$$

This inequality implies that every $B \in \mathcal{M}_{ba}(\Sigma, \mathcal{L}_0(Y,X))$ defines a bounded, hence continuous, linear functional on $\mathcal{S}_\infty(I, \mathcal{L}_1(X,Y))$. Since $\mathcal{S}_\infty(I, \mathcal{L}_1(X,Y))$ is dense in $\mathcal{B}_\infty(I, \mathcal{L}_1(X,Y))$, the functional $\ell_B$ admits a continuous extension to all of $\mathcal{B}_\infty(I, \mathcal{L}_1(X,Y))$. We denote this extension also by the same symbol $\ell_B$. From this we conclude that every $B \in \mathcal{M}_{ba}(\Sigma, \mathcal{L}_0(Y,X))$ defines a continuous linear functional on $\mathcal{B}_\infty(I, \mathcal{L}_1(X,Y))$ and we have

$$\tag{11} \mathcal{M}_{ba}(\Sigma, \mathcal{L}_0(Y,X)) \hookrightarrow \left(\mathcal{B}_\infty(I, \mathcal{L}_1(X,Y))\right)^*.$$ 

Thus it follows from (7) and (11) that

$$\left(\mathcal{B}_\infty(I, \mathcal{L}_1(X,Y))\right)^* \hookrightarrow \mathcal{M}_{ba}(\Sigma, \mathcal{L}_0(Y,X))$$

is an isometric isomorphism. This completes the proof. \[\blacksquare\]
Remark 3.2. Theorem 3.1 easily extends to separable Banach spaces \( \{X, Y\} \) with separable duals \( \{X^*, Y^*\} \) possessing (normalized) biorthogonal bases \( \{x_i, x^*_i\} \) and \( \{y_i, y^*_i\} \).

Remark 3.3. Theorem 3.1 also holds for any compact Polish space \( \mathcal{S} \) replacing the interval \( I \) and furnished with the \( \sigma \)-algebra of Borel sets.

4. Application to Stochastic Control

For any Hilbert space \( X \), we let \( \mathcal{L}_s(X) \) denote the class of bounded self-adjoint operators in \( X \) and \( \mathcal{L}_s^+(X) \) those members of \( \mathcal{L}_s(X) \) which are positive. Similarly \( \mathcal{L}_t(X) \) and \( \mathcal{L}_t^+(X) \) will denote the space of nuclear and positive nuclear operators in \( X \) respectively. Let \( U \) be another separable Hilbert space, where the Brownian motion \( W \equiv \{W(t), t \geq 0\} \) takes values from. Consider the following stochastic system,

\[
\frac{dx}{dt} = Axdt + B(dt)x(t-)+\sigma(t)dW(t),\quad x(0) = x_0, t \in I,
\]

defined on \( X \), where \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( S(t), t \geq 0, \) on \( X \), and \( B \in M_{sa}(\Sigma, \mathcal{L}(X)) \) is an operator-valued measure and \( \sigma \) is a bounded operator-valued function with values in \( \mathcal{L}(U, X) \), and \( W \) is the cylindrical Brownian motion taking values from \( U \). Without loss of generality, in this application, we have restricted our attention to operator-valued measures with values in \( \mathcal{L}(U, X) \), and \( W \) is the cylindrical Brownian motion taking values from \( U \). Without loss of generality, in this application, we have restricted our attention to operator-valued measures with values in \( \mathcal{L}(U, X) \) (possibly a larger space). Let \( (\Omega, \mathcal{F}, \mathcal{F}_t_{t \geq 0}, P) \) denote a complete filtered probability space furnished with a (nondecreasing right continuous with left limits) filtration \( \mathcal{F}_t \subset \mathcal{F} \) to which the Brownian motion is adapted. The system (12) can be interpreted as a hybrid system where the system operator may change abruptly because of natural causes or because of human intervention shutting down certain parts of the system or switching on standby subsystems. This model is also applicable to mechanical structures having flexible appendages, such as communication satellites with flexible antennas, assembly robots with long arms which are controlled by abrupt withdrawal or insertion of arms. Such perturbations can be modeled by the system (12) through the operator-valued measure \( B \) [3-5]. Another example is a Quantum mechanical system [4] where \( A \) is given by a 2 \( \times \) 2 matrix of the ground state Hamiltonian \( H_0 \) as follows,

\[
A \equiv \begin{pmatrix}
0 & H_0 \\
-H_0 & 0
\end{pmatrix}.
\]
In this case, the state space is given by $X = H \times H$ where $H$ is a separable Hilbert space. For details see example (E2) of [4]. It is well known that $A$ generates a unitary group on $X$. The operator-valued measure $B$ is given by

$$B(\sigma) = \begin{pmatrix} 0 & V(\sigma) \\ -V(\sigma) & 0 \end{pmatrix},$$

where $V$ is the potential induced by a laser beam interacting with the atom. Thus for such systems, $B$ can be considered as a control and one of our objectives is to choose this in order to minimize fluctuation caused by the presence of noise.

Equation (12) is said to have a mild solution if the following integral equation

$$(13) \quad x(t) = S(t)x_0 + \int_0^t S(t-s)B(ds)x(s-) + \int_0^t S(t-s)\sigma(s)dW(s), \quad t \in I,$$

has a solution. The solution of this equation is given explicitly by

$$(14) \quad x(t) = U_B(t,0)x_0 + \int_0^t U_B(t,s)\sigma(s)dW(s), \quad t \in I,$$

where $U_B(t,s), 0 \leq s \leq t \leq T$, is the evolution operator determined by the pair $\{A, B(\cdot)\}$. It follows from [5, Theorem 4.1] that if $B \in M_{ca}(\Sigma, \mathcal{L}(X))$, then the pair $\{A, B\}$ generates a strongly measurable evolution operator which is bounded on the triangle $\Delta \equiv \{(s, t) \in I \times I : 0 \leq s \leq t \leq T < \infty\}$. Using the same approach, we can prove the following result.

**Lemma 4.1.** Consider the system

$$(15) \quad dx(t) = Ax(t)dt + B(dt)x(-), x(s-) = \xi, \quad t \in (s, T]$$

for $s \in (0, T]$ and $\xi \in X$ and suppose $A$ is the generator of a $C_0$-semigroup in $X$ and $B \in M_{ba}(\Sigma, \mathcal{L}(X))$. Then, there exists a unique strongly measurable and bounded evolution operator $U_B$ such that the above equation has a unique mild solution $x \in B_\infty((s, T], X)$ given by

$$x(t) = U_B(t, s)\xi, \quad t \in (s, T].$$

**Proof.** See [5, Theorem 4.1].
Remark 4.2. Since $B$ is assumed to have a bounded variation on $I$, it may have at most a countable set of atoms and hence the solution $x$ can have at most a countable set of simple discontinuities. For example, if $\tau$ is an atom, we have

$$U_B(\tau+, s) = U_B(\tau, s) = (I + B(\{\tau\}))U_B(\tau-, s).$$

Using this evolution operator, we can express the solution of the stochastic system (12), or equivalently (13), in the form given by (14) with the mean given by

$$Ex(t) \equiv \bar{x}(t) \equiv U_B(t, 0)\bar{x}_0.$$  

Consider the solution process $x$ given by (14). Let $P_0$ denote the covariance operator corresponding to the initial state $x_0$, and $P(t)$ that corresponding to $x(t), t \in I$, defined by the following identities,

$$(P_0, \zeta) = E(x_0 - \bar{x}_0, \zeta)^2, (P(t), \zeta) = E(x(t) - \bar{x}(t), \zeta)^2, \zeta \in X.$$

Lemma 4.3. Suppose the assumptions of Lemma 4.1 hold, $\{x_0, W\}$ are mutually (statistically) independent, $E|x_0|^2_X < \infty$, and the covariance operator $Q$ of the Brownian motion $\{W(t), t \geq 0\}$, given by $E(W(t), u)^2 = t(Qu, u)$, is a bounded positive self-adjoint operator in $U$, and that $\hat{Q} \in L_1(I, L^+_1(X))$ where $\hat{Q}(t) \equiv \sigma(t)Q\sigma^*(t)$. Then $P \in B_{\infty}(I, L^+_1(X))$ and satisfies the following differential equation in $L(X)$ (in the mild sense)

$$dP(t) = (AP(t) + P(t)A^*)dt + (B(dt)P(t-))$$  

$$+ P(t-)(dt) + \hat{Q}(t)dt, t \in I, P(0) = P_0.$$  

Proof. Define the process $e$ by $e(t) = x(t) - \bar{x}(t), t \geq 0$, and $e_0 \equiv x_0 - \bar{x}_0$. By virtue of Lemma 4.1, it follows from equation (14) that the process $e$ is given by

$$e(t) = U_B(t, 0)e_0 + \int_0^t U_B(t, s)\sigma(s)dW(s), t \in I.$$  

Under the assumption of independence, it follows from this that, for each $\zeta \in X$,
Topological dual of $B_\infty(I, L_1(X,Y))$ with applications

$$E(e(t), \zeta)^2 = E \left\{ (U_B^*(t,0)\zeta, e_0)^2 \right\}$$

$$+ \int_0^t (\hat{Q}(t)U_B^*(t,s)\zeta, U_B^*(t,s)\zeta) ds, \ t \in I.$$  \hspace{1cm} (18)

Thus, by definition of $P(t)$ and $P_0$, this is equivalent to

$$\langle P(t)\zeta, \zeta \rangle = \langle U_B(t,0)P_0U_B^*(t,0)\zeta, \zeta \rangle$$

$$+ \int_0^t \langle U_B(t,s)\hat{Q}(s)U_B^*(t,s)\zeta, \zeta \rangle ds, \ t \in I.$$  \hspace{1cm} (19)

This is the weak form of the mild solution $P$ given by

$$P(t) = U_B(t,0)P_0U_B^*(t,0) + \int_0^t U_B(t,s)\hat{Q}(s)U_B^*(t,s)ds.$$  \hspace{1cm} (20)

Now, replacing $\zeta$ in (19) by $x_i$ from the orthonormal basis $\{x_i\}$ of $X$, and summing over all $i \in N$, we obtain

$$\text{Tr}P(t) = \text{Tr}(U_B(t,0)P_0U_B^*(t,0)) + \int_0^t \text{Tr}(U_B(t,s)\hat{Q}(s)U_B^*(t,s)) ds.$$  \hspace{1cm} (21)

By Lemma 4.1, for each $B \in M_0(\Sigma, L(X))$ the evolution operator $U_B$ is bounded on the triangle $\Delta$, and hence there exists a finite positive number $b$ such that

$$\sup \{ \| U_B(t,s) \|_{L(X)}, 0 \leq s \leq t \leq T \} \leq b.$$  

Since $P_0$ and $\hat{Q}(t)$ are symmetric positive nuclear operators, it follows from (20) that $P(t)$ is also positive self-adjoint and hence from (21), we have

$$\text{Tr}P(t) \leq b^2 \text{Tr}(P_0) + b^2 \int_0^t \text{Tr}(\hat{Q}(s)) ds, t \in I.$$  \hspace{1cm} (22)

By our assumption, $P_0 \in L_1^+(X)$ and $\hat{Q} \in L_1(I, L_1^+(X))$ and consequently it follows from the above inequality that $P \in B_\infty(I, L_1^+(X))$. This completes the proof.
**Problem Statement:** As discussed in the introduction of this section, our problem is to control (minimize) volatility or fluctuation caused by the presence of noise. It is the fluctuation of the process $x(t), t \geq 0$, around its mean $\bar{x}(t), t \geq 0$, which is of concern here. This means that we must minimize the trace of $P(t)$ over the interval $I$ while keeping the variation of the control measure finite.

We can formulate this as an optimal control problem: Consider the system (16) and let $\Gamma \subset M_{ba}(\Sigma, \mathcal{L}(X))$ denote the set of admissible controls (finitely additive bounded operator-valued measures) and introduce the objective (cost) functional,

$$ J(B) = \int_I Tr(\Lambda(t)P(t))dt + \int_I Tr(L(t)B(dt)), $$

where $P$ is the solution of equation (16) corresponding to $B \in \Gamma$, and $\Lambda \in B_\infty(I, \mathcal{L}_+^+(X))$ and $L \in B_\infty(I, \mathcal{L}_1(X))$ are chosen according to physical objectives (design goals). Clearly, the first integral is the standard Lebesgue integral, and the last integral is defined in the sense of duality introduced in Section 3, Theorem 3.1. The problem is: find (or characterize) $B \in \Gamma$ that minimizes the cost functional (23) subject to the dynamic constraint (16).

For other cost functionals see the expression (28) and Remark 4.6.

First, we consider the question of existence of an element $B_o \in \Gamma$ that minimizes the cost functional (23). We call this optimal policy (or optimal structural control).

**Theorem 4.4 (Existence).** Consider the system (16) with the cost functional (23) and suppose the assumptions of Lemma 4.3 hold, $\Lambda \in L_1(I, \mathcal{L}_1^+(X))$ and $L \in B_\infty(I, \mathcal{L}_1(X))$ and the set $\Gamma$ is a weakly sequentially compact subset of $M_{ba}(\Sigma, \mathcal{L}(X))$. Then there exists an optimal policy $B_o \in \Gamma$ minimizing the cost functional (23).

**Proof.** Since $\Gamma$ is weakly sequentially compact and by Eberlein-Smulian theorem it is weakly compact, it suffices to prove that $B \rightarrow J(B)$ is weakly sequentially continuous in the sense that as $B_n \overset{w}{\rightarrow} B_o$ in $M_{ba}(\Sigma, \mathcal{L}(X))$, $J(B_n) \rightarrow J(B_o)$ as $n \rightarrow \infty$. Suppose $B_n \in \Gamma$ and $B_n \overset{w}{\rightarrow} B_o$ in $M_{ba}(\Sigma, \mathcal{L}(X))$. Since $\Gamma$ is closed $B_o \in \Gamma$. Let $P_n$ denote the solution of equation (16) corresponding to $B_n$ and $P_o$ its solution corresponding to $B_o$. In view of the objective functional (23) with $\Lambda \in B_\infty(I, \mathcal{L}_1^+(X))$ and $L \in B_\infty(I, \mathcal{L}_1(X))$, it suffices to verify that, along a subsequence if necessary, $P_n \rightarrow P_o$ in the uniform topology of the space $B_\infty(I, \mathcal{L}_1(X))$. Using the semigroup $S(t), t \geq 0,$
we can write the solution of equation (16) as the solution of the Volterra integral equation

$$P(t) = S(t)P_0S^*(t) + \int_0^t \{ S(t-s)(B(ds)P(s-)+P(s-)B^*(ds))S^*(t-s) \}$$

$$+ \int_0^t S(t-s)\hat{Q}(s)S^*(t-s)ds, \ t \in I,$$

(24)

on the Banach algebra $\mathcal{L}(X)$. Using this equation corresponding to $B_o$ and $B_n$ respectively and taking the difference we have the expression

$$P_o(t) - P_n(t) = E_n(t) + \int_0^t S(t-s)B_n(ds)(P_o - P_n)(s-)S^*(t-s)$$

$$+ \int_0^t S(t-s)(P_o - P_n)(s-)B_n^*(ds)S^*(t-s), \ t \in I,$$

(25)

where $E_n$ is given by

$$E_n(t) = \int_0^t S(t-s)(B_o - B_n)(ds)P_o(s-)S^*(t-s)$$

$$+ \int_0^t S(t-s)P_o(s-)(B_o^* - B_n^*)(ds)S^*(t-s), \ t \in I.$$

(26)

Define

$$e_n(t) \equiv \sup\{|\text{Tr}(P_o(s) - P_n(s))|, 0 \leq s \leq t\}.$$

Recall that the norm of any bounded linear operator coincides with that of its adjoint and hence the total variation norm of a bounded linear operator-valued measure also coincides with that of its adjoint. Using this fact and taking the trace of either side of (25) and using the definition of $e_n$ and generalized Gronwall Lemma [6, Lemma 2, p. 257] which holds also for finitely additive measures having a bounded variation, we obtain the following inequality

$$0 \leq e_n(t) \leq \sup\{|\text{Tr}(E_n(t))|, t \in I\} \exp 2M^2\{ \| B_n \| \}$$

where $\| B_n \|$ denotes the total variation norm of $B_n$ and $M \equiv \sup\{ \| S(t) \|_{\mathcal{L}(X)}, \ t \in I \}$. Since the set $\Gamma \subset M_{ba}(\Sigma, \mathcal{L}(X))$ is bounded, it follows that there
exists a constant $c > 0$ such that

$$0 \leq e_n(t) \leq c \sup\{|Tr(E_n(t))|, t \in I\}. \quad (27)$$

Recall that the semigroup $S(t), t \in I$, is bounded and $P_0(t)$ is nuclear for each $t \in I$, and $B_n \overset{w}{\to} B_0$. Since weak convergence implies weak star convergence, it follows readily from (26) that $Tr(E_n(t)) \to 0$ for each $t \in I$. By computing the trace of either side of (26) it follows from straightforward computation that

$$\sup\{|Tr(E_n(t))|, t \in I, n \in N\} \leq 4M^2 \gamma \| P_0 \|_{B_1(I, L_1(X))}$$

where $\gamma$ is the (total variation) norm bound of the set $\Gamma \subset M_{ba}(\Sigma, L(X))$ and $M$ is the norm bound of the semigroup on the interval $I \equiv [0, T]$. Thus $Tr(E_n(t))$ is uniformly bounded on $I$. This, combined with the facts that $I$ is a compact interval and $Tr(E_n(t)) \to 0$ pointwise on $I$, implies that $Tr(E_n(t)) \to 0$ uniformly on the interval $I$. Hence, it follows from (27) that $e_n(t) \to 0$ for every $t \in I$. In other words, $P_n \to P_0$ in the uniform topology of $B_1(I, L_1(X))$. This proves that $B \to J(B)$ is weakly sequentially continuous and therefore it attains its minimum on $\Gamma$ which is a weakly compact subset of $M_{ba}(\Sigma, L(X))$. This completes the proof of existence of an optimal policy.

Another cost functional which has a significant practical interest is given by

$$J(B) = \int_I Tr(\lambda(t)P(t)) \, dt + \| B \| \quad (28)$$

where $\lambda$ is a nonnegative bounded measurable function which can be chosen so as to assign different weights for different intervals of time as required, and $\| B \|$ is the total variation norm of $B$. The objective is to minimize the cumulative fluctuation while keeping the control energy as small as possible.

**Corollary 4.5.** Consider the system (16) with the cost functional (28) and suppose the assumptions of Lemma 4.3 hold and $\Gamma$ is a weakly compact subset of $M_{ba}(\Sigma, L(X))$ and $\lambda$ is a nonnegative bounded measurable real valued function. Then, there exists an optimal policy $B_o \in \Gamma$ minimizing the functional (28).
Proof. It follows from the proof of Theorem 4.4, that the map

\[ B \rightarrow \int_I \text{Tr}(\lambda(t)P(t))dt \]

is weakly continuous and hence the first term of the cost functional (28) is weakly continuous. It is well known that, in any Banach space, the norm is weakly lower semicontinuous. Thus the last term of (28) is weakly lower semicontinuous and hence the sum is weakly lower semi continuous. By assumption \( \Gamma \) is weakly compact. Hence \( J \) attains its minimum on \( \Gamma \) giving the optimal policy.

\[ \]  

Remark 4.6. The reader can easily verify that, under the same assumptions as given in Theorem 4.4, the following problems have optimal solutions

\[ J(B) \equiv \text{Tr}(P(T)) \rightarrow \inf, \quad J(B) \equiv \int_I \text{Tr}(P(t)) \nu(dt) + (1/2) \| B \|_2^2 \rightarrow \inf. \]

where \( \nu \) is any countably additive bounded positive measure having a bounded total variation on \( I \).

Given the existence of an optimal policy, the next natural question is to find ways to determine it. In general this is done by characterizing the optimal control through optimality (minimum) principle also known as necessary conditions of optimality. We prove the following optimality principle.

Theorem 4.7 (Necessary Conditions of Optimality). Consider the system (16) and the cost functional (23) and suppose the assumptions of Lemma 4.3 hold, \( \Lambda \in L_1(I, \mathcal{L}_1^+(X)) \) and \( L \in B_\infty(I, \mathcal{L}_1(X)) \) and the set \( \Gamma \) is a closed convex and weakly compact subset of \( M_{ba}(\Sigma, \mathcal{L}(X)) \). Then, for \( B_0 \in \Gamma \) to be optimal, it is necessary that there exists a \( Q_0 \in B_\infty(I, \mathcal{L}_1^+(X)) \) satisfying the inequality (29) and the differential equations (30)–(31) on the Banach algebra \( \mathcal{L}(X) \) as stated below,

\[
dJ(B_0, B - B_0) = \int_I \text{Tr}(Q_0(B - B_0)(dt)P_o + P_o(B^* - B_0^*)(dt)Q_o) \\
+ \int_I \text{Tr}(L(B - B_0)(dt) \geq 0 \forall B \in \Gamma,
\]

where \( P_o, Q_o \) are the mild solutions of the following equations,

\[
dP_o = (AP_o + P_oA^*)dt + (B_o(dt)P_o(t-)) + P_o(t-)(B_o^*(dt)) + \dot{Q}(t)dt, \]

\[ P_o(0) = P_0, \]

(30)
\[-dQ_o = (Q_o A + A^*Q_o)dt + (Q_o(t+)B_o(dt) + B_o^*(dt)Q_o(t+)) + \Lambda(t)dt,\]

(31) \[Q_o(T) = 0.\]

**Proof.** We present a brief outline of the proof. Suppose \(B_o \in \Gamma\) is optimal and \(B \in \Gamma\) is arbitrary. Since \(\Gamma\) is a closed convex set, we have \(B_\varepsilon = \frac{B_o + \varepsilon(B - B_o)}{1} \in \Gamma\) for all \(\varepsilon \in [0, 1]\). Let \(P_\varepsilon\) and \(P_o\) denote the solutions of equation (16) corresponding to \(B_\varepsilon\) and \(B_o\), respectively. Then, it follows from the first variation of \(J(B)\) around the optimal \(B_o\) that the Gateaux gradient of the functional \(J(B)\) (see (23)) at \(B_o\) in the direction \(B - B_o\) must satisfy the following inequality,

(32) \[dJ(B_o, B - B_o) = \int_I Tr(\Lambda \hat{P})dt + \int_I Tr(L(B - B_o)(dt)) \geq 0, \forall B \in \Gamma,\]

where \(\hat{P}\) satisfies (in the mild sense) the following differential equation on \(\mathcal{L}(X)\),

\[d\hat{P} = (A\hat{P} + \hat{P} A^*)dt + (B_o(dt)\hat{P} + \hat{P} B_o^*(dt)) + (B - B_o)(dt)P_o + P_o(B - B_o)^*(dt), \quad \hat{P}(0) = 0.\]

(33) It is clear from the above equation that the map \((B - B_o)P_o + P_o(B - B_o)^* \rightarrow \hat{P}\) is a bounded linear map in \((B - B_o)\) from \(M_{ba}(\Sigma, \mathcal{L}(X))\) to \(B_\infty(I, \mathcal{L}_1(X))\). Since \(\Lambda \in L_1(I, \mathcal{L}_s(X)) \subseteq L_1(I, \mathcal{L}(X))\), the map

\[\hat{P} \rightarrow \int_I Tr(\Lambda \hat{P})dt\]

is a continuous linear functional on \(B_\infty(I, \mathcal{L}_1(X))\). Recalling that \(P_o \in B_\infty(I, \mathcal{L}_1^+(X)) \subseteq B_\infty(I, \mathcal{L}_1(X))\), it follows from the above facts that the composition map

\[\begin{align*}
(B - B_o)P_o + P_o(B - B_o)^* \rightarrow \hat{P} \rightarrow \int_I Tr(\Lambda \hat{P})dt
\end{align*}\]

is actually a continuous linear functional on \(M_{ba}(\Sigma, \mathcal{L}_1(X))\). Hence, there exists a \(Q_o \in B_\infty(I, \mathcal{L}_s(X)) \subseteq B_\infty(I, \mathcal{L}(X))\) such that

(34) \[\int_I Tr(\Lambda \hat{P})dt = \int_I Tr\left(Q_o(B - B_o)(dt)P_o + P_o(B - B_o)^*(dt)Q_o\right).\]
Later we show that $Q_\circ \in B_\infty(I, L_1^+(X))$. Since $Q_\circ$ is a uniformly bounded operator-valued function, and $P_\circ$ is a nuclear operator-valued function, and \{\mathcal{B}, \mathcal{B}_\circ\} are elements of $M_{ba}(\Sigma, \mathcal{L}(X))$, it follows from our representation theorem (Theorem 3.1) that the integral on the righthand side of the identity (34) is well defined. Now, integrating the variation $d(Tr(Q_\circ(t)\tilde{P}(t)))$ over the interval I with the terminal condition $Q_\circ(T) = 0$ and using (33) it follows from simple computations that

\[
0 = \int_I Tr\left( [dQ_\circ + (Q_\circ A + A^*Q_\circ)dt + (Q_\circ B_\circ(dt) + B_\circ^*(dt)Q_\circ)]\tilde{P} \right) \\
+ \int_I Tr\left( Q_\circ(B - B_\circ)(dt)P_\circ + P_\circ(B - B_\circ)^*(dt)Q_\circ \right).
\]

Thus, if we choose $Q_\circ$ as being the mild solution (if one exists) of the following evolution equation on the Banach algebra $\mathcal{L}(X)$,

\[
dQ_\circ + (Q_\circ A + A^*Q_\circ)dt + (Q_\circ B_\circ(dt) + B_\circ^*(dt)Q_\circ) = -\Lambda dt, \quad Q_\circ(T) = 0,
\]

it follows from the identity (35) that

\[
\int_I Tr(\Lambda \tilde{P})dt = \int_I Tr\left( Q_\circ(B - B_\circ)(dt)P_\circ + P_\circ(B - B_\circ)^*(dt)Q_\circ \right).
\]

This is precisely the expression (34) as foreseen. Substituting the expression (37) in the inequality (32) we obtain

\[
dJ(B_\circ, B - B_\circ) = \int_I Tr\left( Q_\circ(B - B_\circ)(dt)P_\circ + P_\circ(B - B_\circ)^*(dt)Q_\circ \right) \\
+ \int_I Tr(L(B - B_\circ)(dt)) \geq 0, \quad \forall \ B \in \Gamma.
\]

This proves the necessary inequality (29). From equation (36), it follows that $Q_\circ$, whose existence is foreseen in (34), is given by its mild solution. Equation (36) is precisely the adjoint equation (31) as stated. Considering the question of existence of solution of this equation (adjoint equation), since $B_\circ \in \Gamma \subset M_{ba}(\Sigma, \mathcal{L}(X))$, it follows from Lemma 4.1 that there exists a unique (strongly measurable) bounded evolution operator $U_\circ(t, s) \equiv U_{B_\circ}(t, s), 0 \leq s < t \leq T$, corresponding to the pair \{A, B_\circ\}. 


Hence, equation (36) has a unique mild solution given by

\[
Q_o(t) = \int_t^T U^*_o(T,s) \Lambda(s) U_o(T,s) ds, \quad t \in I,
\]

satisfying the terminal condition \( Q_o(T) = 0 \). Since \( \Lambda \) is a positive self-adjoint operator-valued function, it follows from the above expression that \( Q_o(t) \) is positive self-adjoint, and that

\[
\| Q_o(t) \|_{\mathcal{L}(X)} \leq b^2 \int_t^T \| (\Lambda(s)) \|_{\mathcal{L}(X)} \, ds.
\]

Since by our assumption \( \Lambda \in L_1(I,\mathcal{L}_+^+(X)) \), it follows from the above inequality that \( Q_o \in B_\infty(I,\mathcal{L}_1^+(X)) \). Equation (30) is the state equation corresponding to \( B_o \) and hence we have all the necessary conditions as stated. This completes the brief outline of our proof.

**Remark 4.8.** If in the above theorem we take \( \Lambda \in L_1(I,\mathcal{L}_+^+(X)) \), then the adjoint equation (31) has a mild solution \( Q_o \in B_\infty(I,\mathcal{L}_1^+(X)) \). Again, this follows from (39). Interestingly, this assumption introduces a symmetry in the optimality conditions. Both the state and the adjoint equations (30) and (31) (respectively) have solutions in \( B_\infty(I,\mathcal{L}_1^+(X)) \) contained in the same function space \( B_\infty(I,\mathcal{L}_1(X)) \), having identical regularities.

**Remark 4.9.** The results of this section are also valid for systems subject to Poisson jump processes. Let \( Z \) be a separable Hilbert space and define \( Z_0 = Z \setminus \{0\} \). Consider the system given by the following stochastic evolution equation,

\[
dx = A x dt + B(\sigma(t)) x + \int_{Z_0} C(t) q(dt \times dz), \quad x(0) = x_0, \ t \geq 0,
\]

where \( q \) is a centered Poisson (counting) measure on the product sigma algebra \( \Sigma \times \mathcal{B}(Z_0) \) satisfying

\[
E\{q(\sigma \times \Gamma)\} = 0 \quad \text{and} \quad E\{q^2(\sigma \times \Gamma)\} = \ell(\sigma) m(\Gamma)
\]

for any \( \sigma \in \Sigma \) and \( \Gamma \in \mathcal{B}(Z_0) \) with \( \ell \) denoting the Lebesgue measure on \( \Sigma \). The measure \( m \), called the Lévy measure, is a countably additive bounded positive measure on \( \mathcal{B}(Z_0) \) satisfying

\[
\int_{Z_0} (\eta, z)^2 m(dz) \equiv (Q_m \eta, \eta)_Z < \infty, \ \forall \ \eta \in Z.
\]
For any $\Gamma \in B(Z_0)$, the quantity $m(\Gamma)$ gives the average number jumps of size $\Gamma$ per unit time. Define the operator-valued function $\hat{Q}_m(t), t \geq 0$, by $\hat{Q}_m(t) \equiv C(t)Q_mC^*(t), t \geq 0$, and suppose that it is nuclear. In equation (16), replace $\hat{Q}(t)$ by $\hat{Q}(t) + \hat{Q}_m(t)$. With this modification, all the results of this section, related to the system (12), remain valid for the system (40) under the assumption that the random processes $\{x_0, W, q\}$ are mutually statistically independent.

Before we complete this section, we wish to mention that the necessary conditions of optimality for the problems stated in Remark 4.6 can be developed following a similar procedure as given in Theorem 4.7. For the first problem, set $L = 0$ in equation (29), and $Q_o(T) = I_X$ (identity operator) for the adjoint evolution (31). In this case, $Q_o \in B_{\infty}(I, L^*_1(X))$, not necessarily nuclear unless $U(T, t)$ is Hilbert-Schmidt. This gives all the necessary conditions of optimality.

The second problem, containing $\varphi(B) \equiv (1/2) \| B \|^2$ in the cost functional, is more subtle. Formally, the subdifferential of $\varphi$ is the duality map $\Phi : M^*_ba(\Sigma, L(X)) \rightarrow (M^*_ba(\Sigma, L(X)))^*$ given by

$$\Phi(B) \equiv \left\{ C \in (M^*_ba(\Sigma, L(X)))^* : \int_I Tr(CB(dt)) = \| C \|^2 - \| B \|^2 \right\}.$$  

By virtue of Hahn-Banach theorem, this is nonempty. Thus the necessary inequality for the optimality conditions in the form of (29) is given by

$$dJ(B_o, B - B_o) = \int_I Tr(\tilde{P}(t))\nu(dt) + \int_I Tr(C(t)(B - B_o)(dt))$$

$$= \int_I Tr\left(Q_o(B - B_o)(dt)P_o + P_o(B - B_o)^*(dt)Q_o\right)$$

$$+ \int_I Tr(C(B - B_o)(dt)) \geq 0, \quad \forall \ C \in \Phi(B_o) \equiv \partial \varphi(B_o), \quad \forall \ B \in \Gamma.$$  

The structure of the dual of $M^*_ba(\Sigma, L(X))$ seems to be unknown. However, it follows from the canonical embedding of any Banach space $(Z \hookrightarrow Z^{**})$ that $B_{\infty}(I, L^*_1(X)) \subset (M^*_ba(\Sigma, L(X)))^*$. Thus the above inequality holds for all $C \in \Phi(B_o) \cap B_{\infty}(I, L^*_1(X))$. The associated adjoint equation is given by

$$-dQ_o = (Q_oA + A^*Q_o)dt + (Q_o(t)B_o(dt) + B_o^*(dt)Q_o(t)) + I_X\nu(dt),$$

$$Q_o(T) = 0,$$
with \( Q_o(t) = \int_I^T U_o^*(T, s)U_o(T, s)\nu(ds), t \in I \). Thus we have all the necessary conditions of optimality. Note that in this case also \( Q_o(t) \) is not necessarily nuclear.

**Remark 4.10.** It would be interesting to extend the results presented above to systems of the form

\[
\dot{x} = Ax + Bu + C(x)u + \sigma(t)dW(t), \quad x(0) = x_0, \quad t \in I,
\]

where \( u \in M_{ba}(\Sigma, U) \) with \( U \) being any real Hilbert space and \( C \in \mathcal{L}(X, \mathcal{L}(U, X)) \). One may wish to steer the mean of the process \( x \) along a given path and control its fluctuation around the mean by choosing a pair \( (B_o, u_o) \in \Gamma \times V \subset M_{ba}(\Sigma, \mathcal{L}(X)) \times M_{ba}(\Sigma, U) \) that minimizes an objective (cost) functional \( J(B, u) \) that includes the functional (23) and, in addition, a measure of divergence from a desired trajectory and the cost of control \( u \).

5. **Computational algorithm for two Special cases**

The minimum principle given by Theorem 4.6, specially inequality (29), simplifies significantly in some special cases. For illustration we consider two special cases.

**Case A.** Let \( \{C_i, 1 \leq i \leq n\} \subset \mathcal{L}(X) \) be a given (fixed) family of bounded linear operators and \( M_o \) a weakly compact convex subset of the space of bounded finitely additive vector measures \( M_{ba}(\Sigma, R^n) \). An element \( \mu \in M_{ba}(\Sigma, R^n) \) is an \( n \)-vector of scalar valued measures \( \mu \equiv \{\mu_i, 1 \leq i \leq n\} \). Furnished with the total variation norm, \( \|\mu\| = |\mu|_{ba} \), this is a Banach space. For the admissible set of operator-valued measures, we take the set

\[
\Gamma = \{B \in M_{ba}(\Sigma, \mathcal{L}(X)) : B(\cdot) = \sum_{i=1}^n C_i \mu_i(\cdot), \mu \in M_o\}.
\]

Here, the objective is to choose an element \( \mu^o \in M_o \) that minimizes the cost functional (23) subject to the dynamic constraint (16). In this case, the inequality (29) takes the form

\[
dJ(B_o, B - B_o) \equiv dJ(\mu^o, \mu - \mu^o)
\]

\[
= \int_I \sum_{i=1}^n F_i(P_o, Q_o)(t)(\mu_i(dt) - \mu_i^o(dt)) \geq 0, \quad \forall \mu \in M_o
\]
where

\[ F_i(P_o, Q_o) \equiv Tr(Q_o C_i P_o + P_o C_i^* Q_o + LC_i), \quad i = 1, 2, \ldots, n. \]

Equation (42) can be written as a duality pairing in the spaces \( B_1(I, R^n) \) and \( M_{ba}(\Sigma, R^n) \) as follows,

\[
dJ(\mu^o, \mu - \mu^o) = \int_I (F(P_o, Q_o)(t), (\mu(dt) - \mu^o(dt))) R^n
\]

\[
\equiv < f_o, \mu - \mu^o > \geq 0, \quad \forall \mu \in M_o
\]

where \( f_o \equiv F(P_o, Q_o) = (F_i(P_o, Q_o), i = 1, 2, \ldots, n) \) with \( \{P_o, Q_o\} \) being the mild solutions of equations (30) and (31), respectively, corresponding to the vector measure \( \mu^o \in M_o \) or equivalently, the operator-valued measure \( B_o(\cdot) = \sum_{i=1}^n C_i \mu_i^o(\cdot) \). By Lemma 4.3 and Theorem 4.6, \( P_o \in B_{\infty}(I, L_1^+(X)) \) and \( Q_o \in B_{\infty}(I, L_1^+(X)) \) and by assumption \( L \in B_{\infty}(I, L_1(X)) \), and \( C_i \in L(X) \). Consequently, \( f_o \in B_{\infty}(I, R^n) \). Using the duality map, \( \Phi : B_{\infty}(I, R^n) \to M_{ba}(\Sigma, R^n) \), given by

\[
\Phi(f) \equiv \{ \mu \in M_{ba}(\Sigma, R^n) : < f, \mu > = \| f \|^2 = \| \mu \|^2 \}
\]

for \( f \neq 0 \), one can construct an algorithm for computing the optimal measure \( \mu^o \) iteratively. In general, the duality map \( \Phi \) is multivalued. Now, we are ready to state the steps briefly. Suppose at the n-th stage \( \mu^n \in M_o \) is known, then

**Step 1.** Compute the solutions of equations (30) and (31) replacing \( B_o \) by \( B_n \in \Gamma \), equivalent to \( \mu^n \in M_o \), giving \( \{P_n, Q_n\} \) and compute the gradient \( f_n \equiv F(P_n, Q_n) \).

**Step 2.** Construct

\[
\mu^{n+1} = \mu^n - \varepsilon \nu^n, \quad \text{for any } \nu^n \in \Phi(f_n),
\]

and choose \( \varepsilon > 0 \) sufficiently small so that \( \mu^{n+1} \in M_o \).

**Step 3.** Compute the cost functional corresponding to \( \mu^{n+1} \) giving

\[
J(\mu^{n+1}) = J(\mu^n) + < DJ(\mu^n), \mu^{n+1} - \mu^n > + o(\| \mu^{n+1} - \mu^n \|)
\]

\[
= J(\mu^n) + < f_n, \mu^{n+1} - \mu^n > + o(\| \mu^{n+1} - \mu^n \|)
\]

\[
= J(\mu^n) - \varepsilon < f_n, \nu^n > + o(\varepsilon) = J(\mu^n) - \varepsilon \| f_n \|^2 + o(\varepsilon).
\]
Clearly, it follows from step (3) that the algorithm will reduce the cost at every stage provided \( \varepsilon \) is chosen sufficiently small and positive. Hence, the algorithm is conditionally convergent and leads to a local minimum.

**Case B.** Let \( \mu \in M_{ba}^+(\Sigma) \) (finitely additive positive measures having bounded variation) be a fixed element and let \( B_1(\mathcal{L}(X)) \) denote the closed unit ball of \( \mathcal{L}(X) \). For the admissible set \( \Gamma \) we choose

\[
\Gamma = \{ B \in M_{ba}(\Sigma, \mathcal{L}(X)) : B(\cdot) = C \mu(\cdot), C \in B_1(\mathcal{L}(X)) \}.
\]

This is a weakly sequentially compact subset of \( M_{ba}(\Sigma, \mathcal{L}(X)) \). This can be easily verified by use of the linear functional \( F \) given by,

\[
F(C) \equiv \int_I < g(t), C f(t) > \mu(dt), C \in B_1(\mathcal{L}(X)),
\]

for \( f, g \in B_\infty(I, X) \), and the Lebesgue dominated convergence theorem, and the fact that \( B_1(\mathcal{L}(X)) \) is compact in the weak operator topology on \( \mathcal{L}(X) \).

It is evident that one can choose any closed bounded convex subset of \( \mathcal{L}(X) \), and not merely \( B_1(\mathcal{L}(X)) \). Thus, by our Theorem 4.4, the functional \( J(B) \) given by (23) attains its minimum on \( \Gamma \) as described above proving existence. Now we can present the necessary conditions of optimality. It follows from the inequality (29) that

\[
dJ(B_o, B - B_o) = dJ(C_o, C - C_o)
\]

\[
\equiv \int_I \text{Tr}(Q_o(C - C_o)P_o + P_o(C^* - C_o^*)Q_o) \mu(dt) + \int_I \text{Tr}(L(C - C_o))\mu(dt) \geq 0 \ \forall \ C \in B_1(\mathcal{L}(X)),
\]

where \( P_o, Q_o \in B_\infty(I, L_1(X)) \) are the mild solutions of the following evolution equations on the Banach algebra \( \mathcal{L}(X) \),

\[
dP_o = (AP_o + P_oA^*)dt + (C_oP_o(t-) + P_o(t-)C_o^*)\mu(dt) + \dot{Q}(t)dt,
\]

\[
P_o(0) = P_0,
\]

\[
-dQ_o = (Q_oA + A^*Q_o)dt + (Q_o(t+)C_o + C_o^*Q_o(t+))\mu(dt) + \Lambda(t)dt,
\]

\[
Q_o(T) = 0.
\]
Using these necessary conditions we can now present an algorithm for the determination of the optimal $C_\alpha$. For this, again we need a duality map from the Banach space $L^1(X)$ to its dual $(L^1(X))^* \equiv L(X)$. Let us denote this duality map by $\Psi : L^1(X) \longrightarrow (L^1(X))^* = L(X)$. This is a multivalued map and, for each $D(\neq 0) \in L^1(X)$, it is given by

\begin{equation}
\Psi(D) \equiv \{ C \in L(X) : Tr(CD) = \| C \|^2_{L^1(X)} = \| D \|^2_{L^1(X)} \}.
\end{equation}

By virtue of Hahn-Banach theorem, this is a nonempty set. For example,

\[ C \equiv \| D \|_1 \sum_{i \geq 1} \text{sign}(Dx_i, x_i) \quad x_i \otimes x_i \in \Psi(D), \]

where $\text{sign}(\alpha) = 1$ if $\alpha > 0$ and $-1$ if $\alpha < 0$ and $0$ if $\alpha = 0$. It follows from the expression (44) that, for the fixed $\mu$, there exists an operator $DJ(C_\alpha) \in L^1(X)$ such that

\begin{equation}
dJ(C_\alpha, C - C_\alpha) \equiv Tr(DJ(C_\alpha)(C - C_\alpha)) \geq 0, \quad \forall C \in B_1(L(X)).
\end{equation}

By examining the expression (44), the reader can easily identify this operator as being the integral of some operator-valued functions with respect to the measure $\mu$. Using the necessary condition (44) in this form, we can now present the algorithm. Suppose at the $n$-th stage we know $C_n \in B_1(L(X))$ and now we want to find the next approximation $C_{n+1}$.

**Step 1.** Solve equations (45)–(46) by replacing $C_\alpha$ by $C_n$ to obtain the corresponding solutions $\{P_n, Q_n\}$ in place of $\{P_\alpha, Q_\alpha\}$. Compute the operator $DJ(C_n)$ and note that it depends only on the solutions $\{P_n, Q_n\}$. The gradient at this point is given by

\begin{equation}
dJ(C_n, C - C_n) \equiv Tr(DJ(C_n)(C - C_n)), \quad C \in B_1(L(X)).
\end{equation}

**Step 2.** Construct $C_{n+1}$ as follows

\[ C_{n+1} = C_n - \varepsilon K_n, \quad K_n \in \Psi(DJ(C_n)), \quad \varepsilon > 0, \]

and choose $\varepsilon > 0$ sufficiently small so that $C_{n+1} \in B_1(L(X))$.

**Step 3.** Replacing $C$ by $C_{n+1}$ in the expression (49), and computing the cost functional at $C_{n+1}$, we have
\[ J(C_{n+1}) = J(C_n) + \text{Tr}(DJ(C_n)(C_{n+1} - C_n)) + o(C_{n+1} - C_n) \]
\[ = J(C_n) - \varepsilon \text{Tr}(DJ(C_n)K_n) + o(\varepsilon) \]
\[ = J(C_n) - \varepsilon \| DJ(C_n) \|_{L_1(X)}^2 + o(\varepsilon) \]
\[ = J(C_n) - \varepsilon \| K_n \|_{L_1(X)}^2 + o(\varepsilon). \]

If a stopping criterion is met you are done, otherwise repeat the process with \( C_n \) replaced by \( C_{n+1} \).

Clearly, it follows from step (3) that the algorithm will reduce the cost functional at every stage provided \( \varepsilon \) is chosen sufficiently small and positive. Hence, the algorithm is conditionally convergent leading to a local minimum.

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References


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