RATES OF CONVERGENCE OF CHLODOVSKY-KANTOROVICH POLYNOMIALS IN CLASSES OF LOCALLY INTEGRABLE FUNCTIONS

Paulina Pych-Taberska
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Umultowska 87, 61-614 Poznań, Poland
e-mail: ppych@amu.edu.pl

Dedicated to Professor Michał Kisielewicz on his 70th birthday

Abstract

In this paper we establish an estimation for the rate of pointwise convergence of the Chlodovsky-Kantorovich polynomials for functions $f$ locally integrable on the interval $[0, \infty)$. In particular, corresponding estimation for functions $f$ measurable and locally bounded on $[0, \infty)$ is presented, too.

Keywords and phrases: Chlodovsky polynomial, Kantorovich polynomial, rate of convergence.

2000 Mathematics Subject Classification: 41A25.

1. Introduction

Let $f$ be a function defined on the interval $[0, \infty)$ and let $N = \{1, 2, \ldots\}$. The Bernstein-Chlodovsky polynomials $C_n f$ of the function $f$ are defined as

$$C_n f(x) := \sum_{k=0}^{n} f \left( \frac{kb_n}{n} \right) P_{n,k} \left( \frac{x}{b_n} \right) \quad \text{for} \quad x \in [0, b_n], \quad n \in N,$$  

where $P_{n,k}(t) := \left(\frac{n}{k}\right) t^k (1-t)^{n-k}$ for $t \in [0, 1]$ and $(b_n)$ is a positive increasing sequence satisfying the properties

$$\lim_{n \to \infty} b_n = \infty, \quad \lim_{n \to \infty} \frac{b_n}{n} = 0.$$
These polynomials were first introduced by I. Chlodovsky in 1937 as a generalization of the classical Bernstein polynomials

\[ B_n f(x) := \sum_{k=0}^{n} f \left( \frac{k}{n} \right) P_{n,k}(x), \quad 0 \leq x \leq 1, \]

of functions \( f \) defined on the interval \([0, 1]\) (see [5] or [8], Chap. II). The well-known Chlodovsky theorem states that if

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq b_n} |f(t)| \exp \left( -\alpha \frac{n}{b_n} \right) = 0 \quad \text{for every} \quad \alpha > 0,
\]

then \( \lim_{n \to \infty} C_n f(x) = f(x) \) at every point \( x \) of continuity of \( f \). In 1960 J. Albrecht and J. Radecki [1] proved the Voronovskaya-type theorem for operators (1). Some other approximation properties of the Chlodovsky polynomials can be found e.g. in [3, 7].

For functions \( f \) Lebesgue-integrable on the interval \([0, 1]\) the classical Kantorovich polynomial of order \( n \) is defined as

\[
B^*_n f(x) := \frac{d}{dx} B_{n+1} F(x) \equiv (n + 1) \sum_{k=0}^{n} P_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) \, dt, \quad 0 \leq x \leq 1,
\]

where \( F \) is an indefinite integral of \( f \). It is well known that \( \lim_{n \to \infty} B^*_n f(x) = f(x) \) at any point \( x \) of \((0, 1)\) where \( f \) is the derivative of its indefinite integral (see e.g. [8], Chap. II).

In this paper we consider the Kantorovich-type modification of the Chlodovsky operators (1). Namely, assuming that \( f \in L_{loc}[0, \infty) \), that is \( f \) is locally integrable on \([0, \infty)\), and denoting

\[ F(x) = \int_0^x f(t) \, dt \quad \text{for} \quad x > 0, \]

we define the Chlodovsky-Kantorovich polynomial of degree \( n - 1 \) as

\[ K_{n-1} f(x) := \frac{d}{dx} C_n F(x), \quad n \in N. \]

It is easy to verify that

\[
K_{n-1} f(x) = \frac{n}{b_n} \sum_{k=0}^{n-1} P_{n-1,k} \left( \frac{x}{b_n} \right) \int_{\frac{k b_n}{n}}^{\frac{(k+1) b_n}{n}} f(t) \, dt, \quad 0 \leq x \leq b_n,
\]
(see [3], Section 4).

In order to formulate our first result let us consider those points \( x \in (0, \infty) \) at which
\[
\lim_{h \to 0} \frac{1}{h} \int_0^h (f(x + t) - f(x)) \, dt = 0
\]
and let us introduce the pointwise characteristic
\[
w_x(\delta; f) := \sup_{0 < |h| \leq \delta} \left| \frac{1}{h} \int_0^h (f(x + t) - f(x)) \, dt \right|, \quad \delta > 0.
\]
Clearly, \( w_x(\delta; f) \) is a non-decreasing function of \( \delta > 0 \) and \( \lim_{\delta \to 0^+} w_x(\delta; f) = 0 \) almost everywhere on \([0, \infty)\), that is at every point \( x \in (0, \infty) \) at which (5) is satisfied.

**Theorem 1.** Let \( f \in L_{\text{loc}}[0, \infty) \) and let at a fixed point \( x \in (0, \infty) \) condition (5) be fulfilled. Then, for all integers \( n \) such that \( b_n > 2x, \sqrt{n/b_n} \geq 3 \), we have
\[
|K_{n-1} f(x) - f(x)|
\leq c(q) \left( 1 + \frac{\varphi_n^{q/2}(x)}{x^q} \right) \left( \frac{b_n}{n} \right)^{q-1} \sum_{k=1}^{\lfloor n/b_n \rfloor} k^{q/2} w_x \left( \frac{x}{\sqrt{k}} \right) f(x) \]
\[
+ c(r) \left( \frac{b_n}{n} \right)^{r/2} |f(x)| \]
\[
+ c(r) \left( \frac{b_n}{n} \right)^{r/2} \sqrt{\frac{x}{x(b_n - x)}} \varphi_n^{r/2}(x) \left( \frac{b_n}{n} \right)^{r-1} \int_0^{b_n} |f(t)| \, dt \exp \left(-\frac{n x}{8b_n} \right),
\]
where \( q, r \) are arbitrary positive integers, \( c(q) \) and \( c(r) \) are positive numbers depending only on the indicated parameter \( q \) and \( r \), respectively, \( \varphi_n(x) = x(1 - \frac{x}{b_n}) + \frac{b_n}{n} \) and \( \lfloor n/b_n \rfloor \) denotes the greatest integer not greater than \( n/b_n \).

Taking into account fundamental assumptions (2) and choosing in Theorem 1, \( q = 3, r = 2 \) we easily get

**Corollary 1.** If \( f \in L_{\text{loc}}[0, \infty) \) and if
\[
\lim_{n \to \infty} \int_0^{b_n} |f(t)| \, dt \exp \left(-\alpha \frac{n}{b_n} \right) = 0 \quad \text{for every} \quad \alpha > 0,
\]
then
\[
\lim_{n \to \infty} K_{n-1} f(x) = f(x) \quad \text{almost everywhere on } [0, \infty).
\]

Now, let us consider the subclass \( M_{loc}[0, \infty) \) consisting of all measurable functions \( f \) locally bounded on \([0, 1)\). In this case
\[
w_x(\delta; f) \leq \text{osc}(f; I_x(\delta)) \equiv \sup_{u, v \in I_x(\delta)} |f(u) - f(v)|,
\]
where \( 0 \leq \delta \leq x, \ I_x(\delta) := [x - \delta, x + \delta] \).

**Theorem 2.** Let \( f \in M_{loc}[0, \infty) \) and let at a fixed point \( x \in (0, \infty) \) the one-sided limits \( f(x+) \), \( f(x-) \) exist. Then, for all integers \( n \) such that \( b_n > 2x \), \( \sqrt{n/b_n} \geq 3 \), we have

\[
|K_{n-1} f(x) - \frac{1}{2} (f(x+) + f(x-))| \leq c(q) \left( 1 + \frac{\varphi_n^{\frac{3}{2}}(x)}{x^q} \right) \left( b_n \frac{\varphi_n(b_n)}{n} \sum_{k=1}^{n/b_n} k^{\frac{q-1}{2}} \text{osc} \left( g_x \left( \frac{x^q}{k} \right) \right) \right)
+ \sqrt{\frac{b_n}{x(b_n - x)}} \varphi_n^{\frac{1}{2}}(x) M(b_n; f) \exp \left( - \frac{nx}{8b_n} \right)
+ 2 \sqrt{\frac{b_n}{n}} \sqrt{\frac{b_n}{x(b_n - x)}} \left| f(x+) - f(x-) \right|,
\]

where \( M(b_n; f) = \sup_{0 \leq t \leq b_n} |f(t)| \), \( \varphi_n(x) = x \left( 1 - \frac{x}{b_n} \right) + \frac{b_n}{n} \),

\[
g_x(t) := \begin{cases} 
  f(t) - f(x+) & \text{if } t > x, \\
  0 & \text{if } t = x, \\
  f(t) - f(x-) & \text{if } 0 \leq t < x,
\end{cases}
\]

\( q \) is an arbitrary positive integer, \( c(q) \) is a positive constant depending only on \( q \) and \( c \) is a positive absolute constant.
The function $g_x$ is continuous at $x$. Hence $\lim_{x \to 0^+} \text{osc}(g_x; I_x(\delta)) = 0$. Consequently, Theorem 2 yields the following

**Corollary 2.** If $f \in M_{\text{loc}}[0, \infty)$ and if at a fixed point $x \in (0, \infty)$ the limits $f(x^+), f(x^-)$ exist, then under the Chlodovsky assumption (3), we have

$$\lim_{n \to \infty} K_{n-1} f(x) = \frac{1}{2} (f(x^+) + f(x^-)).$$

(8)

**Remark.** In particular, let us consider the class $BV_\Phi[0, \infty)$ of functions of bounded variation in the Young sense on the interval $[0, \infty)$ (for the definition see e.g. [4, 10]). If $f \in BV_\Phi[0, \infty)$, then $M(b_n; f) \leq M (M = \text{const})$. The estimation given in Theorem 2 and the relation (8) hold true at every point $x \in (0, \infty)$.

## 2. Auxiliary results

We now present certain results which will be used in the proof of our main theorems. For this, let us introduce the notation: given any fixed $x \in [0, b_n]$ and any non-negative integer $q$, we will write

$$\mu_{n,q}(x) := \sum_{k=0}^{n} \left( \frac{kb_n}{n} - x \right)^q P_{n,k} \left( \frac{x}{b_n} \right),$$

$$|\mu_{n,q}(x)| := \sum_{k=0}^{n} \left| \frac{kb_n}{n} - x \right|^q P_{n,k} \left( \frac{x}{b_n} \right).$$

Moreover, we will use the notation $c_j(p), j = 1, 2, \ldots$, for positive constants, not necessarily the same at each occurrence, depending only on the indicated parameter $p$.

**Lemma 1.** Let $n \in N$, $x \in [0, b_n]$.

(i) $\mu_{n,0}(x) = 1, \mu_{n,1}(x) = 0, \mu_{n,2}(x) = \frac{b_n}{n} x \left( 1 - \frac{x}{b_n} \right)$.

(ii) If $s \in N, n \geq 2$, then

$$\mu_{n,2s}(x) \leq c_1(s) \left( \frac{b_n}{n} \right)^s x \left( 1 - \frac{x}{b_n} \right) \left( x \left( 1 - \frac{x}{b_n} \right) + \frac{b_n}{n} \right)^{s-1}.$$
(iii) If $q \in N$, $q \geq 2$, $n \geq 2$, then
\[ |\mu_{n,q}(x)| \leq c_2(q) \left( \frac{b_n}{n} \right)^{\frac{q}{2}} x \left( 1 - \frac{x}{b_n} \right) \left( x \left( 1 - \frac{x}{b_n} \right) + \frac{b_n}{n} \right)^{\frac{q}{2} - 1}. \]

**Proof.** Formulas (i) follow by easy calculation. Suppose $s > 1$ and put $y := x/b_n$. Then $y \in [0, 1]$ and
\[
\mu_{n,2s}(x) = \left( \frac{b_n}{n} \right)^{2s} \sum_{k=0}^{n} (k - ny)^{2s} P_{n,k}(y).
\]
Applying the known representation formula for the above sum (see [6], Lemma 3.6 with $c = -1$) we obtain
\[
\mu_{n,2s}(x) = \left( \frac{b_n}{n} \right)^{2s} \sum_{j=1}^{s} \beta_{j,s} (ny(1-y))^j,
\]
where $\beta_{j,s}$ are real numbers independent of $y$ and bounded uniformly in $n$. Now, let us observe that for $y \in [0, \frac{1}{n}]$ or $y \in [1 - \frac{1}{n}, 1]$ one has $ny(1-y) \leq \frac{n-1}{n} < 1$ and
\[
\left| \sum_{j=1}^{s} \beta_{j,s}(ny(1-y))^j \right| \leq ny(1-y) \sum_{j=1}^{s} |\beta_{j,s}|.
\]
If $y \in [\frac{1}{n}, 1 - \frac{1}{n}]$ then $(ny(1-y))^{-1} \leq \frac{n}{n-1} \leq 2$ and
\[
\left| \sum_{j=1}^{s} \beta_{j,s}(ny(1-y))^j \right| \leq (ny(1-y))^s \sum_{j=1}^{s} |\beta_{j,s}| (ny(1-y))^{j-s} \leq (ny(1-y))^s \sum_{j=1}^{s} 2^{s-j} |\beta_{j,s}|.
\]
Consequently, for all $y \in [0, 1]$ (that is for all $x \in [0, b_n]$) we have
\[
\mu_{n,2s}(x) \leq c_1(s) \left( \frac{b_n}{n} \right)^{2s} n y(1-y) (1 + ny(1-y))^{s-1}
\]
with
\[
c_1(s) \geq \sum_{j=1}^{s} 2^{s-j} |\beta_{j,s}|.
\]
Inequality (ii) follows by taking $y = x/b_n$. The same estimation holds true for $|\mu_{n,q}(x)|$ with even $q$ ($q = 2s$). If $q$ is odd ($q = 2s + 1$), then

$$|\mu_{n,q}(x)| \leq (\mu_{n,4s}(x))^\frac{1}{2}(\mu_{n,2}(x))^\frac{1}{2}$$

by Cauchy-Schwarz inequality, and the proof is complete.

**Lemma 2.** If $n \in N$, $0 < x < b_n$, then

$$\frac{x}{b_n} \left(1 - \frac{x}{b_n}\right) K_{n-1} f(x)$$

(9)

$$= \frac{n}{b_n^2} \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right) P_{n,k} \left(\frac{x}{b_n}\right) \int_{0}^{\frac{kb_n}{n} - x} f(x + t) dt.$$

**Proof.** By (4) and by partial summation, we find that

$$K_{n-1} f(x) = \frac{n}{b_n} \sum_{k=0}^{n-1} P_{n-1,k} \left(\frac{x}{b_n}\right) \int_{0}^{\frac{(k+1)b_n}{n}} f(t) dt =$$

$$= \frac{n}{b_n} P_{n-1,n-1} \left(\frac{x}{b_n}\right) \int_{0}^{b_n} f(t) dt + \frac{n}{b_n} \sum_{k=1}^{n-1} \left( P_{n-1,k-1} \left(\frac{x}{b_n}\right) - P_{n-1,k} \left(\frac{x}{b_n}\right) \right) \int_{0}^{\frac{kb_n}{n}} f(t) dt.$$

Putting $y = x/b_n$ and observing that

$$y(1 - y) (P_{n-1,k-1}(y) - P_{n-1,k}(y)) = \left(\frac{k}{n} - y\right) P_{n,k}(y)$$

for $k = 1, 2, \ldots, n - 2$ and

$$y(1 - y)n P_{n-1,n-1}(y) = y P_{n,n-1}(y) = n(1 - y) P_{n,n}(y),$$

we easily get

$$\frac{x}{b_n} \left(1 - \frac{x}{b_n}\right) K_{n-1} f(x) = \frac{n}{b_n} \sum_{k=0}^{n} \left(\frac{k}{n} - \frac{x}{b_n}\right) P_{n,k} \left(\frac{x}{b_n}\right) \int_{0}^{\frac{kb_n}{n}} f(t) dt.$$
Now, it is enough to recall that
\[ \sum_{k=0}^{n} \left( \frac{kb_n}{n} - x \right) P_{n,k} \left( \frac{x}{b_n} \right) = \mu_{n,1}(x) = 0 \]

(Lemma 1 (i)). Consequently,
\[ \frac{x}{b_n} \left( 1 - \frac{x}{b_n} \right) K_{n-1} f(x) = \frac{n}{b_n^2} \sum_{k=0}^{n} \left( \frac{kb_n}{n} - x \right) P_{n,k} \left( \frac{x}{b_n} \right) \int_{x}^{\frac{kb_n}{n}} f(t) dt \]
and the proof is complete.

Note that a corresponding representation like in the formula (9) for the classical Kantorovich polynomials is given in [2].

Lemma 3. If \( 0 < \delta \leq x < b_n \) then
\[ \sum_{|\frac{kb_n}{n} - x| \geq \delta} P_{n,k} \left( \frac{x}{b_n} \right) \leq 2 \exp \left( -\frac{n\delta^2}{4xb_n} \right) \]
for all \( n \in N \) such that \( b_n \geq \frac{3x^2}{3\delta - \delta} \).

The proof of Lemma 3 runs as in [1] and is based on the known Chlodovsky inequality ([8], Theorem 1.5.3)):
\[ \sum_{|k-nt| \geq 2z \sqrt{nt(1-t)}} P_{n,k}(t) \leq 2 \exp \left( -z^2 \right), \]
provided that \( 0 \leq t \leq 1, 0 \leq z \leq \frac{3}{2} \sqrt{nt(1-t)} \).

Lemma 4. Let \( 0 < x < b_n \) and let \( n \geq 2 \).
(i) If \( 0 \leq k \leq n - 1 \), then
\[ P_{n-1,k} \left( \frac{x}{b_n} \right) \leq \frac{1}{e} \sqrt{\frac{b_n}{n}} \sqrt{\frac{b_n}{x(b_n - x)}}. \]
(ii) \[
\left| \sum_{\frac{kb_n}{n} < k \leq n} P_{n-1,k} \left( \frac{x}{b_n} \right) - \frac{1}{2} \right| \leq 0.82 \sqrt{2} \sqrt{\frac{b_n}{n}} \sqrt{\frac{b_n}{x(b_n - x)}}.
\]

**Proof.** Estimation (i) follows from the result by X.M. Zeng [11] (Theorem 1): if \(0 \leq k \leq n\) and \(y \in (0, 1)\), then
\[
P_{n,k}(y) \leq \frac{1}{\sqrt{2\pi y}} \frac{1}{\sqrt{n}y(1 - y)}.
\]

Inequality (ii) is an immediate consequence of the Berry-Esseen Theorem:
\[
\left| \sum_{\frac{kb_n}{n} > y} P_{n,k}(y) - \frac{1}{2} \right| < \frac{0.82}{\sqrt{ny(1 - y)}}, \quad 0 < y < 1
\]
(see e.g., [12], Lemma 2).

3. Proofs of theorems

**Proof of Theorem 1.** In view of Lemma 1 (i) one can write
\[
\frac{x}{b_n} \left( 1 - \frac{x}{b_n} \right) f(x) = \frac{n}{b_n^2} \mu_{n,2}(x)f(x)
= \frac{n}{b_n^2} \sum_{k=0}^{n} \left( \frac{kb_n}{n} - x \right) P_{n,k} \left( \frac{x}{b_n} \right) \int_{0}^{\frac{kb_n}{n}} dt f(x).
\]
The above identity and the representation (9) lead to
\[
\frac{x}{b_n} \left( 1 - \frac{x}{b_n} \right) \left( K_{n-1}f(x) - f(x) \right)
= \frac{n}{b_n^2} \sum_{k=0}^{n} \left( \frac{kb_n}{n} - x \right) P_{n,k} \left( \frac{x}{b_n} \right) \int_{0}^{\frac{kb_n}{n} - x} (f(x + t) - f(x)) dt
\equiv \sum_{k \in \Lambda} + \sum_{k \in \Omega}.
\]
where \( \Lambda \) and \( \Omega \) are the sets of indices \( k \in \{0, 1, \ldots, n\} \) such that \( \left| \frac{kb_n}{n} - x \right| \leq x \) and \( \frac{kb_n}{n} - x > x \), respectively.

For the sake of brevity let us introduce the notation: \( d_n = \sqrt{b_n/n} \), \( m = \lfloor \sqrt{n/b_n} \rfloor \), \( w_x(\delta; f) = w_x(\delta) \). Consider the sum \( \sum_{k \in \Lambda} \) and divide the set \( \Lambda \) in the following manner: \( \Lambda = \bigcup_{j=0}^{m} \Lambda_j \), where \( \Lambda_j \) are the sets of indices \( k \) such that

\[
0 \leq \left| \frac{kb_n}{n} - x \right| \leq x d_n \quad \text{if} \quad j = 0, \\
jd_n < \left| \frac{kb_n}{n} - x \right| \leq (j + 1)d_n \quad \text{if} \quad j = 1, 2, \ldots, m - 1, \\
mjd_n < \left| \frac{kb_n}{n} - x \right| \leq x \quad \text{if} \quad j = m.
\]

In view of definition (6),

\[
\left| \sum_{k \in \Lambda} \right| \leq \sum_{j=0}^{m-1} T_{n,j}(x)w_x((j + 1)d_n) + T_{n,m}(x)w_x(x),
\]

where

\[
T_{n,j}(x) := \frac{n}{b_n^2} \sum_{k \in \Lambda_j} \left( \frac{kb_n}{n} - x \right)^2 P_{n,k} \left( \frac{x}{b_n} \right).
\]

From Lemma 1 (i) one has

\[
T_{n,0}(x) \leq \frac{n}{b_n^2} \mu_{n,2}(x) = \frac{x}{b_n} \left( 1 - \frac{x}{b_n} \right).
\]

Next, given any positive integer \( q \), we have

\[
T_{n,j}(x) \leq \frac{n}{b_n^2} \frac{1}{(j+1)d_n^q} \sum_{k=0}^{n} \left| \frac{kb_n}{n} - x \right|^{q+2} P_{n,k} \left( \frac{x}{b_n} \right)
\]

for \( j = 1, 2, \ldots, m \). Hence Lemma 1 (iii) yields

\[
T_{n,j}(x) \leq \frac{c_1(q)}{j^q b_n} \frac{x}{b_n} \left( 1 - \frac{x}{b_n} \right) \varphi_n^{q/2}(x)
\]
where \( \varphi_n(x) = x \left(1 - \frac{x}{b_n}\right) + \frac{b_n - n}{n} \). Consequently,

\[
\left| \sum_{k \in \Lambda} \right| \leq \frac{x}{b_n} \left(1 - \frac{x}{b_n}\right) \left(1 + \frac{c_1(q)}{x^q} r_n^{q/2}(x) \right) \left( \sum_{j=1}^{m-1} \frac{w_x(j + 1) x d_n}{j^q} + \frac{w_x(x)}{m^q} \right).
\]

Clearly,

\[
\sum_{j=1}^{m-2} \frac{w_x((j + 1) x d_n)}{j^q} \leq 3^q d_n^{q-1} \int_{2d_n}^{m d_n} \frac{w_x(x t)}{t^q} \, dt
\]

\[
\leq 3^q d_n^{q-1} \int_1^{m^2} (\sqrt{s})^{q-3} w_x \left( \frac{x}{\sqrt{s}} \right) \, ds
\]

\[
\leq c_2(q) d_n^{q-1} \sum_{k=1}^{m^2-1} (\sqrt{k + 1})^{q-3} w_x \left( \frac{x}{\sqrt{k}} \right)
\]

and

\[
\frac{w_x(x)}{(m-1)^q} + \frac{w_x(x)}{m^q} \leq \frac{2}{(m-1)^q} w_x(x) \leq 3^q d_n^{q-1} w_x(x).
\]

Hence

\[
\left| \sum_{k \in \Omega} \right| \leq c_3(q) \frac{x}{b_n} \left(1 - \frac{x}{b_n}\right) \left(1 + \frac{\varphi_n^{q/2}(x)}{x^q} \right) \left( \frac{q}{n^q} \sum_{k=1}^{m^2-1} (\sqrt{k + 1})^{q-3} w_x \left( \frac{x}{\sqrt{k}} \right) \right).
\]

(11)

Now, let us consider the sum \( \sum_{k \in \Omega} \) in formula (10). Given any positive integer \( r \), we have

\[
\left| \sum_{k \in \Omega} \right| \leq \frac{n}{b_n^2 x^r} \left( \sum_{k \in \Omega} \left| \frac{k b_n}{n} - x \right|^{r+2} P_{n,k} \left( \frac{x}{b_n} \right) \right) |f(x)|
\]

\[
+ \frac{n}{b_n^2 x^r} \sum_{k \in \Omega} \left| \frac{k b_n}{n} - x \right|^{r+1} P_{n,k} \left( \frac{x}{b_n} \right) \int_0^{b_n} |f(t)| \, dt
\]

\[
\leq \frac{n}{b_n^2 x^r} \left| \mu_{n,r+2}(x) \right| |f(x)|
\]

\[
+ \frac{n}{b_n^2 x^r} \int_0^{b_n} |f(t)| \, dt \left( \mu_{n,2r+2}(x) \right)^{1/2} \left( \sum_{k \in \Omega} P_{n,k} \left( \frac{x}{b_n} \right) \right)^{1/2}.
\]
Applying Lemmas 1 and 3 we then get

$$\left| \sum_{k \in \Omega} \right| \leq c_4(r) \frac{x}{x^r} \frac{b_n}{b_n} \left( 1 - \frac{x}{b_n} \right) \left( \frac{b_n}{n} \right)^{r/2} \varphi_n(r/2(x)|f(x)|$$

$$+ c_4(r) \frac{x}{x^r} \frac{b_n}{b_n} \left( 1 - \frac{x}{b_n} \right) \left( \frac{b_n}{n} \right)^{r/2} \frac{b_n}{x(b_n - x)} \varphi_n(r/2(x) \int_0^{b_n} |f(t)|dt \exp \left( - \frac{nx}{8b_n} \right).$$

This gives the desired conclusion when combined with (10) and (11).

**Proof of Theorem 2.** Let \( f \in M_{loc}(0, \infty) \) and let the limits \( f(x+), f(x-) \) exist at a fixed point \( x > 0 \). Consider the function \( g_x \) defined by (7). It is easily seen that

$$f(t) - \frac{f(x+)}{2} + \frac{f(x-)}{2} = g_x(t) + \frac{f(x+)}{2} \text{sgn}_x(t)$$

$$+ \left( f(x) - \frac{f(x+)}{2} + \frac{f(x-)}{2} \right) \delta_x(t),$$

where \( \text{sgn}_x(t) = \text{sgn}(t - x), \delta_x(t) = 1 \) if \( t = x, \delta_x(t) = 0 \) otherwise (see e.g. [9]). Hence

$$K_{n-1}f(x) - \frac{f(x+)}{2} + \frac{f(x-)}{2} = K_{n-1}g_x(x) + \frac{f(x+)}{2} \text{sgn}_x(x).$$

The function \( g_x \) is continuous at \( x \) and \( g_x(x) = 0 \). So, \( K_{n-1}g_x(x) = K_{n-1}g_x(x) - g_x(x) \) can be estimated as in the proof of Theorem 1. Namely, using formula (10) in which \( f \) is replaced by \( g_x \) and observing that

$$w_x(\delta; g_x) \leq \text{osc}(g_x; I_x(\delta)) \quad \text{for} \quad 0 < \delta \leq x$$

we get the estimation for \( \left| \sum_{k \in \Omega} \right| \) as in (11) with \( w_x \left( \frac{x}{\sqrt{k}} \right) \) replaced by \( \text{osc} \left( g_x; I_x \left( \frac{x}{\sqrt{k}} \right) \right) \). Indeed, we estimate the sum \( \sum_{k \in \Omega} \) as follows:

$$\left| \sum_{k \in \Omega} \right| \leq \frac{2n}{b_n^2} M(b_n; f) \sum_{k \in \Omega} \left( k b_n / n - x \right)^2 P_{n,k} \left( x / b_n \right).$$
where $M(b_n; f) = \sup_{0 \leq t \leq b_n} |f(t)|$. Next, the Cauchy-Schwarz inequality and Lemmas 1, 3 lead to

$$
\sum_{k \in \Omega} \left| K_{n-1} g_x(x) \right| \leq 2M(b_n; f) \frac{n}{b_n^2} \left( \frac{\mu_{n,4}(x)}{\varphi_n^{1/2}(x) M(b_n; f)} \right)^{1/2} 
\leq c \frac{\varphi_n^{q/2}(x)}{x^q} \left( 1 - \frac{x}{b_n} \right) \sqrt{\frac{b_n}{x(b_n - x)}} \varphi_n^{1/2}(x) M(b_n; f) \exp \left( -\frac{nx}{8b_n} \right),
$$

where $c$ is an absolute positive constant. Consequently,

$$
|K_{n-1} g_x(x)| \leq c(q) \left( 1 + \frac{\varphi_n^{q/2}(x)}{x^q} \right) \frac{b_n}{n} \sum_{k=1}^{\lfloor n/b_n \rfloor} (\sqrt{k})^{q-3} \text{osc} \left( g_x; I_x \left( \frac{x}{\sqrt{k}} \right) \right)
$$

$$
+ c \frac{b_n}{x(b_n - x)} \varphi_n^{1/2}(x) M(b_n; f) \exp \left( -\frac{nx}{8b_n} \right),
$$

where $q$ is arbitrary positive integer, $c(q)$ is a positive constant depending only on $q$ and $c$ is an absolute constant.

Now it is enough to estimate the term $K_{n-1} \text{sgn}_x(x)$. Choose the integer $l$ such that $x \in \left[ \frac{l}{n} b_n, \frac{l+1}{n} b_n \right)$. It is clear that

$$
K_{n-1} \text{sgn}_x(x) = \sum_{k \geq l} P_{n-1,k} \left( \frac{x}{b_n} \right) - \sum_{k < l} P_{n-1,k} \left( \frac{x}{b_n} \right)
$$

$$
+ \frac{n}{b_n} P_{n-1,l} \left( \frac{x}{b_n} \right) \left( \frac{l}{n} b_n + \frac{b_n}{n} - 2x \right)
$$

$$
= 2 \sum_{k > l} P_{n-1,k} \left( \frac{x}{b_n} \right) - 1 + 2P_{n-1,l} \left( \frac{x}{b_n} \right) \frac{n}{b_n} \left( \frac{l+1}{n} b_n - x \right).
$$

Therefore,

$$
|K_{n-1} \text{sgn}_x(x)| \leq 2 \left| \sum_{k \geq l} P_{n-1,k} \left( \frac{x}{b_n} \right) - \frac{1}{2} \right| + 2P_{n-1,l} \left( \frac{x}{b_n} \right)
$$

$$
\leq 4 \sqrt{\frac{b_n}{n}} \sqrt{\frac{b_n}{x(b_n - x)}},
$$
by Lemma 4. Combining the above estimations for $|K_{n-1}g_x(x)|$ and $|K_{n-1}\text{sgn}_x(x)|$ with (12) we obtain the desired conclusion. Thus the proof of Theorem 2 is complete.

REFERENCES


Received 12 May 2009