Robustness of Estimation
of First-Order Autoregressive Model
Under Contaminated Uniform White Noise

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Abstract

The first-order autoregressive model with uniform innovations is considered. In this paper, we study the bias-robustness and MSE-robustness of modified maximum likelihood estimator of parameter of the model against departures from distribution of white noise. We used the generalized Beta distribution to describe these departures.

Keywords: autoregressive model, bias, MSE, robustness, generalized Beta distribution.


1. Introduction

Consider the following autoregressive model

\[ X_t = \rho X_{t-1} + \varepsilon_t, \quad t = \ldots, -1, 0 + 1, \ldots, \] with \( 0 \leq \rho < 1, \]
where the $\varepsilon_t$’s are i.i.d and distributed according to uniform distribution $U(0, \theta)$.

Bell and Smith (1986) studied the estimation and testing problem on the parameter $\rho$ for model (1) with $\varepsilon_t$ are i.i.d and non-negative. The study is established in three parametric cases: Gaussian, exponential and uniform as well as for the nonparametric case where it only assumed that $\varepsilon$’s have a positive continuous distribution. In these different models, various types of point estimates and procedures of test for $\rho$ are introduced by authors.

In the case of model (1) defined above, Nouali and Fellag (2002) obtained an approximation of bias of maximum likelihood estimator (MLE) for parameter $\rho$. A formula of approximate bias of MLE is given also when a single outlier occurs at specified time with a known amplitude. In the same model, Nouali and Fellag (2005) proposed a new testing procedures which perform a test on the parameter $\rho$ in presence of single innovation outlier comparing to those proposed by Bell and Smith (1986).

Now, we assume $X_0$ distributed as $U(0, \theta/(1 - \rho))$ and observe the segment of observations

$$X_1, X_2, \ldots, X_n, \quad n \text{ fixed}$$

from a model (1). The maximum likelihood estimator (MLE) for $\rho$ is (Bell and Smith, 1986) $\hat{\rho} = \min_{2 \leq t \leq n} (X_t/X_{t-1})$. Then, we can write

$$\hat{\rho} = \rho + \min_{2 \leq t \leq n} \left( \frac{\varepsilon_t}{X_{t-1}} \right).$$

Since the process is mean stationary with mean $m_\theta = \frac{\theta}{2(1-\rho)}$, we can use the method proposed by Andel (1988) in exponential model which consist to substitute $m_\theta$ for $X_{t-1}$ in (3). Then, the estimator becomes $\hat{\rho} = \rho + \frac{1}{m_\theta} \min_{2 \leq t \leq n} (\varepsilon_t)$.

Let $b(\rho)$ and $MSE(\rho)$ be the bias and the mean square error of estimator $\hat{\rho}$ in model (1) respectively. Hence,
Robustness of estimation of first-order autoregressive ...

\[ b(\rho) = \frac{1}{m_{\theta}} E \left( \min_{2 \leq t \leq n} (\varepsilon_t) \right) = \frac{2}{n} (1 - \rho), \quad \text{(Nouali and Fellag, 2002)} \]

\[ MSE(\rho) = \frac{1}{m_{\theta}^2} E \left( \min_{2 \leq t \leq n} (\varepsilon_t)^2 \right) = \frac{8}{n(n+1)} (1 - \rho)^2. \]

In fact, the pdf function \( h(\cdot) \) of random variable \( Z = \min_{2 \leq t \leq n} (\varepsilon_t) \) is obtained after simple computation

\[ h(z) = \begin{cases} 
\frac{n-1}{\theta} \left(1 - \frac{z}{\theta}\right)^{-2} & \text{if } 0 \leq z \leq \theta, \\
0 & \text{else.}
\end{cases} \]

with \( E(Z) = \frac{\theta}{n} \) and \( E(Z^2) = \frac{2\theta^2}{n(n+1)} \).

Our aim is to discuss a bias-robustness and MSE-robustness of estimator \( \tilde{\rho} \) when the distribution of white noise is contaminated by generalized Beta distribution. Note that the bias-robustness of estimator \( \tilde{\rho} \) is considered by Fellag and Ibazizen (2001) in the case of contaminated exponential white noise. The authors used a class of exponential power distribution to modeling the departures from original model.

2. Bias and mse of \( \tilde{\rho} \) under generalized beta white noise

The statistical model with uniform white noise is denoted by

\[ M_{1,1} = (\mathbb{R}^+, \mathcal{B}^+, U(0, \theta), \theta > 0), \]

where \( \mathbb{R}^+ \) is the real-half, \( \mathcal{B}^+ \) is the family of Borel subsets of \( \mathbb{R}^+ \).
We can consider the following model as extension of original model \( M_{1,1} \):

\[
M_{\alpha,\beta} = (\mathbb{R}^+, \mathbb{B}^+, \text{GBeta}_\theta(\alpha, \beta), \theta > 0, \alpha, \beta \in \mathbb{R}^+_+),
\]

where \( \text{GBeta}_\theta(\alpha, \beta) \) is a general Beta distribution on three parameters defined on an interval \((0, \theta)\) with density

\[
f_{\alpha,\beta}(x, \theta) = \frac{1}{\theta^{\alpha+\beta-1} B(\alpha, \beta)} x^{\alpha-1}(\theta - x)^{\beta-1}, 0 \leq x \leq \theta.
\]

Notice that the uniform distribution over \((0, \theta)\) is a particular generalized Beta distribution of parameters \( \theta, \alpha \) and \( \beta \) with \( \alpha = \beta = 1 \).

Now, we suppose that the model \( M_{1,1} \) is violated in such way that the innovations \( \varepsilon_1, \ldots, \varepsilon_n \) are distributed according to \( \text{GBeta}_\theta(\alpha, \beta) \) rather than the uniform distribution over \((0, \theta)\) and we assume that \( \theta \) is equal to 1 without loss of generality.

We have

\[
E(\varepsilon_t) = \frac{\alpha}{\alpha + \beta}, \forall t = 1, \ldots, n.
\]

We assume that \( X_0 \) is distributed according to \( \text{GBeta}_1(1, 1) \) with

\[
\tilde{\theta} = \frac{2\alpha}{(\alpha + \beta)(1 - \rho)},
\]

then a process \((X_t)\) is mean stationary with mean

\[
m_{\alpha,\beta} = \frac{\alpha}{(\alpha + \beta)(1 - \rho)}
\]

and we can write
\[
\hat{\rho} = \rho + \frac{1}{m_{\alpha,\beta}} \min_{2 \leq t \leq n} \varepsilon_t = \rho + \frac{(\alpha + \beta)(1 - \rho)}{\alpha} \min_{2 \leq t \leq n} \varepsilon_t.
\]

Let \( H(z) \) and \( h(z) \) be the distribution and the pdf function respectively of random variable \( Z = \min_{2 \leq t \leq n} \varepsilon_t \). After computation, we obtain

\[
h(z) = (n - 1) f_{\alpha,\beta}(z), [1 - F_{\alpha,\beta}(z)]^{n-2}.
\]

The bias and MSE of \( \hat{\rho} \) in Model \( M_{\alpha,\beta} \) are given by numerical formulas (4) and (5) respectively

\[
b_{\alpha,\beta}(\rho) = \frac{(n - 1)(\alpha + \beta)(1 - \rho)}{\alpha B(\alpha, \beta)}
\]

\[
\cdot \int_0^1 z^\alpha (1 - z)^{\beta - 1} [1 - B_z(\alpha, \beta)]^{n-2}. dz, \forall n.
\]

\[
MSE_{\alpha,\beta}(\rho) = \frac{(n - 1)(\alpha + \beta)^2(1 - \rho)^2}{\alpha^2 B(\alpha, \beta)}
\]

\[
\cdot \int_0^1 z^{\alpha+2}, (1 - z)^{\beta - 1} [1 - B_z(\alpha, \beta)]^{n-2}. dz, \forall n.
\]

where
\[ B_z(\alpha, \beta) = \int_0^z x^{\alpha-1} (1 - x)^{\beta-1} dx \]

denotes the incomplete Beta function.

In this paper, we study a behavior of bias and MSE of estimator \( \hat{\rho} \) in the following two particular extensions of original model:

\[ M_\alpha = M_{\alpha,1} \quad \text{and} \quad M_\beta = M_{1,\beta}. \]

We used a measure of robustness in sense of stability proposed by Zielinski (1977). In our case, if a property of estimator is a bias, this measure of robustness takes the following form

\[
rb_{[\lambda_1, \lambda_2]}(\rho) = \sup_{\lambda_1 \leq \lambda < \lambda_2} b(\rho) - \inf_{\lambda_1 \leq \lambda < \lambda_2} b(\rho), \quad \text{in model } M_\lambda
\]

with \( \lambda = \alpha \) or \( \beta \).

This function is called the "bias-robustness function". It represents a maximal oscillation of the bias of estimator \( \hat{\rho} \) in "supermodel" \( M_\lambda \).

- We say that the estimator is "absolutely robust" according to bias criteria in model \( M_\lambda \) if

\[
rb_\lambda(\rho) = 0, \forall \lambda > 0.
\]

- When \( rb_{\lambda \in I_1}(\rho) < rb_{\lambda \in I_2}(\rho) \) then the bias of the estimator is more stable for \( \lambda \in I_1 \) than for \( \lambda \in I_2 \), where \( I_1 \) and \( I_2 \) are two intervals of \( \mathbb{R}_+^* \).

In the same manner as bias-robustness function, we can define MSE-robustness function which denote by \( rm(.) \). For more details on this robustness approach, we refer the reader to Zielinski (1977).
2.1. The behavior of bias and mse of $\hat{\rho}$ in model $M_\alpha$

In the model $M_\alpha$, the density $h(z)$ is written

$$h(z) = \alpha(n - 1).z^{-1}.[1 - z^\alpha]^{n-2}.$$ 

The bias and MSE of estimator $\hat{\rho}$ are given respectively by formula (6) and (7)

(6) \[ b_\alpha(\rho) = \frac{(1 + \alpha)}{\alpha^2} B(1/\alpha, n)(1 - \rho), \forall n. \]

\[ MSE_\alpha(\rho) \]

(7) \[ = \frac{(1 + \alpha)^2}{\alpha^2} (n - 1) B(1 + 2/\alpha, n - 1)(1 - \rho)^2, \forall n. \]

Let us adopt the following notations:

$b_{\text{min}}, M_{\text{min}}$ : the minimal value of bias respectively of MSE.

$b_{\text{max}}, M_{\text{max}}$ : the maximal value of bias respectively of MSE.

In order to illustrate a behavior of bias and MSE of estimator $\hat{\rho}$ in model $M_\alpha$, we give in Tables 1 and 2 the exact values of bias and MSE as function of $\alpha$ and $\rho$ for different smaller values of size ($n=5, 10, \text{and } 20$). The values are computed using a formula (6) and (7).
Table 1. Values of the $b_\alpha(\rho)$ for $n = 5, 10$ and 20.

<table>
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<th>$n = 10$</th>
<th>$n = 20$</th>
</tr>
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<td>$\rho = 0.9$</td>
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<td>0.0331</td>
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<td>0.2400</td>
<td>0.0400</td>
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<td>0.0545</td>
</tr>
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<td>0.0610</td>
</tr>
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<td>0.7121</td>
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</tr>
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<td>0.0899</td>
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<tr>
<td>$b_{max}$</td>
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<td>0.6</td>
<td>0.1</td>
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Table 2. Values of the $MSE_\alpha(\rho)$ for $n = 5, 10$ and 20.

<table>
<thead>
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<th></th>
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<td>0.0000</td>
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<td>0.0000</td>
</tr>
<tr>
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<td>0.0350</td>
<td>0.0010</td>
<td>0.0061</td>
<td>0.0022</td>
<td>0.0001</td>
<td>0.0003</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
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<td>0.0214</td>
<td>0.0077</td>
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<td>0.0009</td>
<td>0.0000</td>
</tr>
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</tr>
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<td>0.1259</td>
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<td>0.01</td>
<td>1</td>
<td>0.36</td>
<td>0.01</td>
<td>1</td>
<td>0.36</td>
<td>0.01</td>
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</table>
The results leads the following comments:

**Comment 1.** One can remark that the bias and MSE of estimator $\tilde{\rho}$ are an increasing functions of $\alpha$. The variation of these two criterions is very small when $\alpha \in (0, 1]$ than when $\alpha \in (1, +\infty]$.

**Comment 2.** When $\alpha$ is close to zero, the bias and MSE of estimator $\tilde{\rho}$ tend to value $b_{\min} = M_{\min} = 0$.

**Comment 3.** When $\alpha$ tend to infinity,

\[
\begin{align*}
    b_\alpha(\rho) &\rightarrow b_{\max} = (1 - \rho), \\
    MSE_\alpha(\rho) &\rightarrow M_{\max} = (1 - \rho)^2.
\end{align*}
\]

After computations, we obtain the following expressions of bias and MSE robustness functions:

\[
\begin{align*}
    rb_{(0,1]}(\rho) &= b_1(\rho) - b_{\min} = \frac{2(1 - \rho)}{n}, \\
    rb_{(1, +\infty)}(\rho) &= b_{\max} - b_1(\rho) = \frac{(n - 2)(1 - \rho)}{n}, \\
    rm_{(0,1]}(\rho) &= MSE_1(\rho) - M_{\min} = \frac{8(1 - \rho)^2}{n(n + 1)}, \\
    rm_{(1, +\infty)}(\rho) &= M_{\max} - MSE_1(\rho) = \frac{n(n + 1) - 8}{n(n + 1)}. \end{align*}
\]
and we can establish the following inequalities

\[ rb_{(0,1)}(\rho) < rb_{(1, +\infty)}(\rho), \forall n. \]

\[ rm_{(0,1)}(\rho) < rm_{(1, +\infty)}(\rho), \forall n \geq 4. \]

The two inequalities show that the estimator \( \hat{\rho} \) is more stable for \( \alpha \in (0, 1] \) than for \( \alpha \in (1, +\infty[ \) with respect to bias and MSE.

### 2.2 The behavior of bias and mse of \( \hat{\rho} \) in model \( M_\beta \)

As in above section, we give the exact formulas of bias and MSE of estimator \( \hat{\rho} \). Also, we derive the expressions of bias-robustness function and MSE-robustness function.

So, in model \( M_\beta \), the density \( h(z) \) is written

\[ h(z) = \beta. (n - 1). |1 - z|^{\beta(n-1)-1}. \]

The expressions of bias and MSE of estimator \( \hat{\rho} \) take the following forms

\[ b_\beta(\rho) = \frac{(1 + \beta)}{1 + \beta(n - 1)}. (1 - \rho), \forall n \geq 1. \]

\[ MSE_\beta(\rho) = \frac{2(1 + \beta)^2}{(1 + \beta(n - 1))(2 + \beta(n - 1))}. (1 - \rho)^2, \forall n \geq 1. \]

We present in Tables 3 and 4 the exact values of bias and MSE respectively of estimator \( \hat{\rho} \) as function of \( \beta \) and \( \rho \) for different smaller values of size \( (n=5, 10, \text{and} 20) \). Here, the values are computed using a formula (8) and (9).
Table 3. Values of the $b_\beta(\rho)$ for $n = 5, 10$ and $20$.

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<td>0.0576</td>
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<td>0.0058</td>
</tr>
<tr>
<td>$b_{\text{min}}$</td>
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<td>0.15</td>
<td>0.025</td>
<td>0.1111</td>
<td>0.0666</td>
<td>0.0111</td>
<td>0.0526</td>
<td>0.0315</td>
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</table>
Table 4. Values of the $MSE_\beta(\rho)$ for $n = 5, 10$ and 20.

<table>
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<th>$\beta$</th>
<th>$n = 5$</th>
<th></th>
<th></th>
<th>$n = 10$</th>
<th></th>
<th></th>
<th>$n = 20$</th>
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<td></td>
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<td>$\rho = 0.4$</td>
<td>$\rho = 0.9$</td>
<td>$\rho = 0$</td>
<td>$\rho = 0.4$</td>
<td>$\rho = 0.9$</td>
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<td>0.1</td>
<td>1</td>
<td>0.6</td>
<td>0.1</td>
<td>1</td>
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<td>0.7829</td>
<td>0.2818</td>
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<tr>
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<td>0.2057</td>
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<td>0.2707</td>
<td>0.0974</td>
<td>0.0027</td>
<td>0.1034</td>
<td>0.0372</td>
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<tr>
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<td>0.1232</td>
<td>0.0034</td>
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<td>0.0389</td>
<td>0.0011</td>
<td>0.0308</td>
<td>0.0111</td>
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<td>0.0042</td>
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<td>0.0859</td>
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<td>0.0262</td>
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<td>0.0022</td>
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<td>0.0193</td>
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<tr>
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<td>0.0720</td>
<td>0.0020</td>
<td>0.0474</td>
<td>0.0171</td>
<td>0.0005</td>
<td>0.0115</td>
<td>0.0042</td>
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<tr>
<td>3</td>
<td>0.1758</td>
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<td>0.0394</td>
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<td>8</td>
<td>0.1444</td>
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<td>0.0014</td>
<td>0.0300</td>
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<td>0.0003</td>
<td>0.0069</td>
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<td>0.0104</td>
<td>0.0003</td>
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<td>0.0024</td>
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<td>$M_{min}$</td>
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<td>0.0450</td>
<td>0.0012</td>
<td>0.0246</td>
<td>0.0088</td>
<td>0.0002</td>
<td>0.0055</td>
<td>0.0019</td>
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</tbody>
</table>
Some properties obtained in the model $M_\beta$ are given in the following comments:

**Comment 1.** The inverse situation is presented in model $M_\beta$ to that observed in model $M_\alpha$. The bias and MSE of estimator $\tilde{\rho}$ are a decreasing functions of $\beta$. The two criterions increase strongly when $\beta \in (0,1]$ but decrease slowly when $\beta \in (1, +\infty[$.

**Comment 2.** When $\beta$ is close to zero, the bias and MSE of estimator $\tilde{\rho}$ tend to value $b_{\text{max}} = M_{\text{max}} = (1 - \rho)$.

**Comment 3.** When $\alpha$ tend to infinity,

\[
\begin{align*}
  b_\alpha(\rho) & \rightarrow b_{\text{min}} = \frac{(1 - \rho)}{n - 1}, \\
  \text{MSE}_\alpha(\rho) & \rightarrow M_{\text{min}} = \frac{2(1 - \rho)^2}{(n - 1)^2}.
\end{align*}
\]

After computations, we obtain the following expressions of bias and MSE robustness functions:

\[
\begin{align*}
  rb_{(0,1]}(\rho) &= b_{\text{max}} - b_1(\rho) = \frac{(n-2)}{n} (1 - \rho), \\
  rb_{(1, +\infty]}(\rho) &= b_1(\rho) - b_{\text{min}} = \frac{(n-2)}{n(n-1)} (1 - \rho), \\
  rm_{(0,1]}(\rho) &= M_{\text{max}} - \text{MSE}_1(\rho) = \frac{n(n+1) - 8}{n(n+1)} (1 - \rho)^2, \\
  rb_{(1, +\infty]}(\rho) &= \text{MSE}_1(\rho) - M_{\text{min}} = \frac{6n^2 - 9n - 4}{(n-1)(n^2 + n)} (1 - \rho)^2.
\end{align*}
\]
Robustness of estimation of first-order autoregressive ...

This leads to

\[ rb_{(1, +\infty)}(\rho) < rb_{(0, 1)}(\rho), \forall n. \]

\[ rm_{(1, +\infty)}(\rho) < rm_{(0, 1)}(\rho), \forall n \geq 4. \]

In this situation, we can say that the estimator \( \hat{\rho} \) is more stable for \( \beta \in (1, +\infty] \) than for \( \beta \in (0, 1] \) with respect to bias and MSE.

3. Conclusions

The bias and the MSE of estimator \( \hat{\rho} \) present a very small oscillation under model \( M_\alpha \) when \( \alpha \in (0, 1] \) than for \( \alpha \in (1, +\infty[ \).

In model \( M_\beta \), we get completely opposite behavior to that obtained in model \( M_\alpha \). We can conclude that the estimator \( \hat{\rho} \) has a good properties (more robust) in model \( M_\alpha \) for \( \alpha \in (0, 1] \) and in model \( M_\beta \) for \( \beta \in (1, +\infty[ \) with respect to bias and mean square error.

References


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