DECOMPOSITION OF COMPLETE BIPARTITE MULTIGRAPHS INTO PATHS AND CYCLES HAVING $k$ EDGES

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Abstract

We give necessary and sufficient conditions for the decomposition of complete bipartite multigraph $K_{m,n}(\lambda)$ into paths and cycles having $k$ edges. In particular, we show that such decomposition exists in $K_{m,n}(\lambda)$, when $\lambda \equiv 0 \pmod{2}$, $m, n \geq \frac{k}{2}$, $m + n > k$, and $k(p + q) = 2mn$ for $k \equiv 0 \pmod{2}$ and also when $\lambda \geq 3$, $\lambda m \equiv \lambda n \equiv 0 \pmod{2}$, $k(p + q) = \lambda mn$, $m, n \geq k$, (resp., $m, n \geq 3k/2$) for $k \equiv 0 \pmod{4}$ (respectively, for $k \equiv 2 \pmod{4}$). In fact, the necessary conditions given above are also sufficient when $\lambda = 2$.

Keywords: path, cycle, graph decomposition, multigraph.

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1. Introduction

Unless stated otherwise all graphs considered here are finite, simple, and undirected. For the standard graph-theoretic terminology the readers are referred to [8]. A cycle of length $m$ is called an $m$-cycle and it is denoted by $C_m$ and a path of length $m$ is called an $m$-path and it is denoted by $P_{m+1}$. A circuit (directed cycle) of length $m$ is called an $m$-circuit and it is denoted by $\overrightarrow{C}_m$. Let $K_m$ denote a complete graph on $m$ vertices, $K_{m,n}$ denote a complete bipartite graph with
m and n vertices in the parts, and \( K_{m,n}^* \) denote a complete bipartite symmetric directed graph with m and n vertices in the parts. A graph whose vertex set is partitioned into sets \( V_1, \ldots, V_m \) such that the edge set is \( \bigcup_{i \neq j \in [m]} V_i \times V_j \) is called a complete m-partite graph denoted by \( K_{n_1, \ldots, n_m} \), where \( |V_i| = n_i \) for all i. For any integer \( \alpha > 0 \), \( \alpha G \) denotes a union of \( \alpha \) edge-disjoint copies of \( G \). The \( \lambda \)-multiplication of \( G \), denoted \( G(\lambda) \), is the multigraph obtained from a graph \( G \) by replacing each edge with \( \lambda \) edges. For a graph \( G \), \( G - I \) denotes the graph \( G \) with a 1-factor \( I \) removed. Let \( x_0x_1 \cdots x_{k-2}x_{k-1} \) and \( (x_0x_1 \cdots x_{k-1}x_0) \) respectively denote the path \( P_k \) and the cycle \( C_k \) with vertices \( x_0, x_1, \ldots, x_{k-1} \) and edges \( x_0x_1, x_1x_2, \ldots, x_{k-2}x_{k-1}, x_{k-1}x_0 \).

By a decomposition of the graph \( G \), we mean a list of edge-disjoint subgraphs of \( G \) whose union is \( G \) (ignoring isolated vertices). For the graph \( G \), if \( E(G) \) can be partitioned into \( E_1, \ldots, E_k \) such that the subgraph induced by \( E_i \) is \( H_i \), for all \( i, 1 \leq i \leq k \), then we say that \( H_1, \ldots, H_k \) decompose \( G \) and we write \( G = H_1 \oplus \cdots \oplus H_k \), since \( H_1, \ldots, H_k \) are edge-disjoint subgraphs of \( G \). For \( 1 \leq i \leq k \), if \( H_i \cong H \), we say that \( G \) has a \( H \)-decomposition. If \( G \) has a decomposition into \( p \) copies of \( H_1 \) and \( q \) copies of \( H_2 \), then we say that \( G \) has a \( \{pH_1, qH_2\} \)-decomposition. If such a decomposition exists for all admissible pairs of \( p \) and \( q \) satisfying trivial necessary conditions, then we say that \( G \) has a full \( \{H_1, H_2\} \)-decomposition or \( G \) is fully \( \{H_1, H_2\} \)-decomposable.

Study on full \( \{H_1, H_2\} \)-decomposition of graphs is not new. Abueida, Daven, and Roblee [1, 3] completely determined the values of \( n \) for which \( K_n(\lambda) \) admits the \( \{pH_1, qH_2\} \)-decomposition such that \( H_1 \oplus H_2 \cong K_t \), when \( \lambda \geq 1 \) and \( |V(H_1)| = |V(H_2)| = t \), where \( t \in \{4, 5\} \). Let \( S_k \) denotes a star on \( k \) vertices, i.e. \( S_k = K_{1,k-1} \). Abueida and Daven [2] proved that there exists a \( \{pK_k, qS_{k+1}\} \)-decomposition of \( K_n \) for \( k \geq 3 \) and \( n \equiv 0, 1 \) (mod \( k \)). Abueida and O’Neil [4] proved that for \( k \in \{3, 4, 5\} \), the \( \{pC_k, qS_k\} \)-decomposition of \( K_n(\lambda) \) exists, whenever \( n \geq k + 1 \) except for the ordered triples \( (k, n, \lambda) \in \{3, 4, 1\}, (4, 5, 1), (5, 6, 1), (5, 6, 2), (5, 6, 4), (5, 7, 1), (5, 8, 1) \}. Abueida and Daven [2] obtained necessary and sufficient conditions for the \( \{pC_4, q(2K_2)\} \)-decomposition of the Cartesian product and tensor product of paths, cycles, and complete graphs. Shyu [17] obtained a necessary and sufficient condition for the existence of a full \( \{P_5, C_4\} \)-decomposition of \( K_n \). Shyu [18] proved that \( K_n \) has a full \( \{P_4, S_4\} \)-decomposition if and only if \( n \geq 6 \) and \( 3(p + q) = \binom{n}{2} \). Also he proved that \( K_n \) has a full \( \{P_k, S_k\} \)-decomposition with a restriction \( p \geq k/2 \), when \( k \) even (resp., \( p \geq k \), when \( k \) odd). Shyu [19] obtained a necessary and sufficient condition for the existence of a full \( \{P_4, K_3\} \)-decomposition of \( K_n \). Shyu [20] proved that \( K_n \) has a full \( \{C_4, S_5\} \)-decomposition if and only if \( 4(p + q) = \binom{q}{2} \), \( q \neq 1 \), when \( n \) is odd and \( q \geq \max\{3, \lceil \frac{q}{4}\rceil \} \), when \( n \) is even. Shyu [21] proved that \( K_{m,n} \) has a full \( \{P_k, S_k\} \)-decomposition for some \( m \) and \( n \) and also obtained some necessary and sufficient condition for the existence of a full \( \{P_4, S_4\} \)-decomposition of
Chou et al. [9] proved that for a given triple \((p, q, r)\) of nonnegative integers, \(G\) decompose into \(p\) copies of \(C_4\), \(q\) copies of \(C_6\), and \(r\) copies of \(C_8\) such that \(4p + 6q + 8r = |E(G)|\) in the following two cases: (a) \(G = K_{m,n}\) with \(m\) and \(n\) both even and greater than four (b) \(G = K_{n,n} - I\), where \(n\) is odd. Chou and Fu [10] proved that the existence of a full \(\{C_4, C_2\}\)-decomposition of \(K_{2u,2v}\), where \(t/2 \leq u, v < t\), when \(t\) even (resp., \((t + 1)/2 \leq u, v \leq (3t - 1)/2\), when \(t\) odd) implies such decomposition in \(K_{2u,2v}\), where \(m, n \geq t\) (resp., \(m, n \geq (3t + 1)/2\)).

The authors [11] reduced the bounds of the sufficient conditions obtained by Chou and Fu [10] for the existence of a full \(\{C_4, C_2\}\)-decomposition of \(K_{2u,2v}\), when \(t > 2\). Lee and Chu [13, 14] obtained a necessary and sufficient condition for the existence of a full \(\{P_k, S_k\}\)-decomposition of \(K_{n,n}\) and \(K_{m,n}\). Lee and Lin [15] obtained a necessary and sufficient condition for the existence of a full \(\{pC_k, qS_{k+1}\}\)-decomposition of \(K_{n,n} - I\). Abueida and Lian [7] obtained necessary and sufficient conditions for the existence of a \(\{pC_k, qS_{k+1}\}\)-decomposition of \(K_n\) for some \(n\). Recently, the authors [12] obtained some necessary and sufficient conditions for the existence of a full \(\{P_{k+1}, C_k\}\)-decomposition of \(K_n\) and \(K_{m,n}\).

In this paper, we study only the existence of a full \(\{P_{k+1}, C_k\}\)-decomposition of \(K_{m,n}(\lambda)\), we abbreviate the notation for such decomposition as \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\). The obvious necessary condition for such existence is \(k(p + q) = |E(K_{m,n}(\lambda))|\). As we consider only cases where all vertices are of even degree, the case \(p \neq 1\) is also obviously necessary, since the presents of a single path in the decomposition would give two vertices of odd degree and the resulting graph is not cycle decomposable. Call the situation with \(k(p + q) = |E(K_{m,n}(\lambda))|\), all vertex degrees are even, and \(p \neq 1\) the good case.

We prove that in the good case \(K_{m,n}(\lambda)\) has a \((k; p, q)\)-decomposition, when \(\lambda \equiv 0\) (mod 2), \(m, n \geq \frac{k}{2}\), \(m + n > k\), and \(k(p + q) = 2mn\) for \(k \equiv 0\) (mod 2). Further, we show that if \(K_{m,n}(\lambda)\), \(\lambda \geq 3, k \equiv 0\) (mod 4) (resp., \(k \equiv 2\) (mod 4)) has a \((k; p, q)\)-decomposition in the good case with \(k/2 \leq m, n \leq k\), (resp., \(k/2 \leq m, n \leq 3k/2\)) then such decomposition also exists in the good case, when \(\lambda \geq 3; m, n \geq k\) (resp., \(m, n \geq 3k/2\)).

To prove our results, we use the following:

**Theorem 1** [12]. Let \(p\) and \(q\) be nonnegative integers and \(k, m, n\) be positive even integers such that \(k \equiv 0\) (mod 4). For \(m \leq n\), the graph \(K_{m,n}\) has a \((k; p, q)\)-decomposition if and only if \(m \geq \frac{k}{2}\), \(n \geq \lceil \frac{k+1}{2}\rceil\), \(k(p + q) = mn\), and \(p \neq 1\).

**Theorem 2** [22]. \(K^*_m\) has a \(\overrightarrow{C_k}\)-decomposition if and only if \(m \geq \frac{k}{2}\), \(n \geq \frac{k}{2}\), and \(k\) divides \(2mn\).

By considering the underlying graph of \(K^*_m\), we have the following from Theorem 2.
Theorem 3. The graph \( K_{m,n}(2) \) has a \( C_k \)-decomposition if and only if \( m \geq \frac{k}{2} \), \( n \geq \frac{k}{2} \), and \( k \) divides \( 2mn \).

2. \((k;p,q)\)-Decomposition of \( K_{m,n}(\lambda) \) when \( k \equiv 0(\text{mod } 2)\)

In this section, we investigate the existence of \((k;p,q)\)-decomposition of \( K_{m,n}(\lambda) \), when \( k \equiv 0(\text{mod } 2) \).

Construction 4. Let \( C_\lambda \) and \( C_\mu \) be two cycles of length \( k \), where \( C_\lambda = (x_1x_2 \cdots x_kx_1) \) and \( C_\mu = (y_1y_2 \cdots y_ky_1) \). If \( v \) is a common vertex of \( C_\lambda \) and \( C_\mu \) such that at least one neighbour of \( v \) from each cycle (say, \( x_i \) and \( y_j \)) does not belong to the other cycle, then we have two edge-disjoint paths of length \( k \), say \( P_\lambda \) and \( P_\mu \) from \( C_\lambda \) and \( C_\mu \) as follows (see Figure 1), where \( P_\lambda = (C_\lambda - vx_i) \cup vy_j, P_\mu = (C_\mu - vy_j) \cup vx_i \).

![Figure 1](image)

Remark 5. Let \( k \in \mathbb{N} \). If \( G \) and \( H \) have a \((k;p,q)\)-decomposition, then \( G \oplus H \) has such a decomposition.

Lemma 6. Let \( p, q \) be nonnegative integers and \( \{k,m,n\} \in \mathbb{N} \) such that \( k \equiv 0(\text{mod } 2) \) and \( m + n > k \). The graph \( K_{m,n}(2) \) has a \((k;p,q)\)-decomposition if and only if \( m, n \geq k/2, k(p+q) = 2mn \), and \( p \neq 1 \).

Proof. Necessity. Conditions \( m, n \geq k/2, k(p+q) = 2mn \), and \( p \neq 1 \) are trivial.

Sufficiency. Let \( k \equiv 0(\text{mod } 2) \). In order to have a \( C_k \)-decomposition in \( K_{m,n}(2) \), we can always find \( u, v \) such that \( k = 2uv, m = ru, n = sv, r \geq v, \) and \( s \geq u \) where \( r \) and \( s \) are positive integers. We denote the vertices of the partite sets of \( K_{ru,sv} \) by \( x_i, 0 \leq i \leq ru - 1 \) and \( y_j, 0 \leq j \leq sv - 1 \). By Theorem 3, the
Now we construct the required number of \( x \)- and \( C \)-indices of \( K \), where the indices of \( x \) are to be taken with modulo \( ru \), and those of \( y \) with modulo \( sv \). Now we construct the required number of \( P_{k+1} \) from the \( C_k \)-decomposition given above, in two cases.

**Case 1:** \( p \) is even. For a fixed \( \mu \) and \( 0 \leq \lambda \leq s - 1 \), we can have \( C_{\lambda \mu} \) and \( C_{(\lambda + 1)\mu} \) as above. Since \( x_{\mu}y_{\lambda v} \in E(C_{\lambda \mu}) \), \( x_{\mu}y_{(\lambda + u + v - 1)} \in E(C_{(\lambda + 1)\mu}) \), \( y_{\lambda v} \notin V(C_{(\lambda + 1)\mu}) \), and \( y_{(\lambda + u + v - 1)} \notin V(C_{\lambda \mu}) \), we have two edge-disjoint paths of length \( k \), say \( P_{\lambda \mu} \) and \( P_{(\lambda + 1)\mu} \) from \( C_{\lambda \mu} \) and \( C_{(\lambda + 1)\mu} \) as follows (see Figure 2).

\[
P_{\lambda \mu} = (C_{\lambda \mu} - x_{\mu}y_{\lambda v}) \cup x_{\mu}y_{(\lambda + u + v - 1)},
\]

\[
P_{(\lambda + 1)\mu} = (C_{(\lambda + 1)\mu} - x_{\mu}y_{(\lambda + u + v - 1)}) \cup x_{\mu}y_{\lambda v}.
\]

Similarly, we can find pairs of paths of length \( k \) from the pairs of cycles \( C_{\lambda \mu} \) and \( C_{(\lambda + 1)\mu} \), where \( 0 \leq \lambda \leq s - 2 \) or \( s - 1 \) and \( 0 \leq \mu \leq r - 1 \). Hence the graph \( K_{m,n}(2) \) has the desired decomposition.

Now for a fixed \( \lambda \) and \( 0 \leq \mu \leq r - 1 \), we can have \( C_{\lambda \mu} \) and \( C_{(\lambda + 1)\mu} \) as above. Since \( x_{\mu p}y_{(\lambda + p)q - 1} \in E(C_{\lambda \mu}) \), \( x_{(\mu + q + 1)p - 1}y_{(\lambda + p)q - 1} \in E(C_{(\lambda + 1)\mu}) \), \( x_{\mu p} \notin V(C_{(\lambda + 1)\mu}) \), and \( x_{(\mu + q + 1)p - 1} \notin V(C_{\lambda \mu}) \), we have two edge-disjoint paths of length \( k \), say \( P_{\lambda \mu} \) and \( P_{\lambda(\mu + 1)} \) from \( C_{\lambda \mu} \) and \( C_{(\lambda + 1)\mu} \) as follows (see Figure 3).

\[
P_{\lambda \mu} = (C_{\lambda \mu} - x_{\mu p}y_{(\lambda + p)q - 1}) \cup x_{(\mu + q + 1)p - 1}y_{(\lambda + p)q - 1},
\]

\[
P_{\lambda(\mu + 1)} = (C_{\lambda(\mu + 1)} - x_{(\mu + q + 1)p - 1}y_{(\lambda + p)q - 1}) \cup x_{\mu p}y_{(\lambda + p)q - 1}.
\]
In Construction 4, we have three edge-disjoint paths of length \( k \), where \( P \) divides 2 or \( \mu \) divides \( C \). Similarly, we can find pairs of paths of length \( k \) from the pairs of cycles \( C_{\lambda \mu} \) and \( C_{\lambda(\mu+1)} \), where \( \mu = 0, 2, \ldots, r-2 \) or \( r-1 \). Hence we have the desired paths.

**Case 2:** \( p \) is odd. Fixing \( v = \gcd(n,k/2) \), we have \( u = k/2v \), \( s = n/v \). Since \( k \) divides \( 2mn \), i.e., \( 2uv \) divides \( 2mn \) and \( v \) divides \( n \), we have \( r = m/u \).

**Subcase 2a:** \( (v + 2)u - 1 \leq m \) and \( v + 2 \leq r \). Since \( r \geq 3 \) and \( s \geq 1 \), we can have \( C_{00}, C_{01}, \) and \( C_{02} \) (see Figure 4). By applying a procedure similar to Construction 4, we have three edge-disjoint paths of length \( k \), say \( P_{00}, P_{01}, \) and \( P_{02} \) from \( C_{00}, C_{01}, \) and \( C_{02} \) as follows (see Figure 5).
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\[ P_{00} = (C_{00} - x_0y_{uv-1}) \cup x_{(v+1)u-1}y_{uv-1}, \]
\[ P_{01} = (C_{01} - x_{(v+1)u-1}y_{uv-1}) \cup x_{(v+2)u-1}y_{uv-1}, \]
\[ P_{02} = (C_{02} - x_{(v+2)u-1}y_{uv-1}) \cup x_0y_{uv-1}. \]

By applying a procedure similar to Case 1, the remaining pairs of cycles \( C_{\lambda \mu} \oplus C_{\lambda(\mu+1)} \), \((\lambda, \mu), (\lambda, \mu + 1) \neq (0,0), (0,1), (0,2)\) decomposes into pairs of paths. Hence the graph \( K_{m,n}^{(2)} \) has the desired decomposition.

Subcase 2b: \((u + 2)v - 1 \leq n \) and \( u + 2 \leq s \). Since \( r \geq 1 \) and \( s \geq 3 \), we can have \( C_{00}, C_{10}, \) and \( C_{20} \) (see Figure 6). By applying a procedure similar to
Construction 4, we have three edge-disjoint paths of length $k$, say $P_{00}$, $P_{10}$, and $P_{20}$ from $C_{00}$, $C_{10}$, and $C_{20}$ as follows (see Figure 7).

$$P_{00} = (C_{00} - x_0y_{uv-1}) \cup x_0y_{(u+1)v-1},$$
$$P_{10} = (C_{10} - x_0y_{(u+1)v-1}) \cup x_0y_{(u+2)v-1},$$
$$P_{20} = (C_{20} - x_0y_{(u+2)v-1}) \cup x_0y_{uv-1}.$$

By applying a procedure similar to Case 1, the remaining pairs of cycles $C_{\lambda\mu} \oplus C_{(\lambda+1)\mu}$ ($\lambda, \mu$), ($\lambda + 1, \mu$) $\neq$ $(0, 0), (1, 0), (2, 0)$ decomposes into pairs of paths. Hence the graph $K_{m,n}(2)$ has the desired decomposition.

Subcase 2c: $(v + 1)u - 1 \leq m$, $(u + 1)v - 1 \leq n$, $u + 1 \leq s$, and $v + 1 \leq r$, $m$ or $n \neq k/2$. Since $r, s \geq 2$ we can have $C_{00}$, $C_{10}$, and $C_{11}$. By applying a procedure similar to Case 1, we have two edge-disjoint paths of length $k$, say $P_{10}$ and $P_{11}$ from $C_{10}$ and $C_{11}$ as follows:

$$P_{10} = (C_{10} - x_0y_{(u+1)v-1}) \cup x_{(v+1)u-1}y_{(u+1)v-1},$$
$$P_{11} = (C_{11} - x_{(v+1)u-1}y_{(u+1)v-1}) \cup x_0y_{(u+1)v-1}.$$

Now consider $C_{00}$ and $P_{11}$ (see Figure 8); since $x_0y_{uv-1} \in E(C_{00})$, $x_{(u+1)v-2}y_{uv-1}$ $\notin V(C_{00})$, and $x_0 \in V(P_{11})$, we have two edge-disjoint paths of length $k$, say $P_{00}$ and $P_{11}$ from $C_{00}$ and $P_{11}$ as follows (see Figure 9).

$$P_{00} = (C_{00} - x_0y_{uv-1}) \cup x_{(v+1)u-2}y_{uv-1},$$
$$P_{11} = (P_{11} - x_{(v+1)u-2}y_{uv-1}) \cup x_0y_{uv-1}.$$

By applying a procedure similar to Case 1, the remaining pairs of cycles both $C_{\lambda\mu} \oplus C_{(\lambda+1)\mu}$ and $C_{\lambda\mu} \oplus C_{(\mu+1)\lambda}$ ($\lambda, \mu$), ($\lambda + 1, \mu$) ($\lambda, \mu + 1$) $\neq$ $(0, 0), (0, 1), (1, 1)$
decomposes into pairs of paths. Hence the graph $K_{m,n}(2)$ has the desired decomposition.

**Subcase 2d:** $m = k/2 + 1$ and $n = k/2$. When $m = k/2 + 1$ and $n = k/2$, we have $s = p = 1$ and $r = q + 1$. Since $\lambda = 2$ and $0 \leq \mu \leq r - 1$, we can have $C_{00}$ and $C_{01}$ (see Figure 10). By applying a procedure similar to Case 1, we have two edge-disjoint paths of length $k$, say $P_{00}$ and $P_{01}$ from $C_{00}$ and $C_{01}$ as follows (see Figure 11).

\[
P_{00} = (C_{00} - x_0y_2a-3) \cup x_2a-2y_2a-3,
\]
\[
P_{01} = (C_{01} - x_2a-2y_2a-3) \cup x_0y_2a-3.
\]
Let \( a = r + 1/2 \). Now we consider \( \mathbb{P}_{00} \) and \( \mathbb{C}_0a \) (see Figure 12). Since \( x_{2a-1}y_{a-2} \in E(C_{a0}) \), \( x_{a-1}y_{a-2} \in E(\mathbb{P}_{00}) \), and \( x_{a-1} \notin V(C_{a0}) \) we have two edge-disjoint paths of length \( k \), say \( \mathbb{P}_{0a} \) and \( \mathbb{P}_{00} \) from \( \mathbb{C}_0a \) and \( \mathbb{P}_{00} \) as follows (see Figure 13).

\[
\begin{align*}
&x_0 \quad y_0 \quad x_1 \quad y_1 \quad x_{2a-4} \quad y_{2a-4} \quad x_{2a-3} \quad y_{2a-3} \quad x_{2a-2} \\
&x_1 \quad y_0 \quad x_2 \quad y_1 \quad x_{2a-3} \quad y_{2a-4} \quad x_{2a-2} \quad y_{2a-3} \quad x_0
\end{align*}
\]

Figure 11. \( \mathbb{P}_{00} \cup \mathbb{P}_{01} \).

\[
\begin{align*}
&x_0 \quad y_0 \quad x_1 \quad y_1 \quad x_{a-2} \quad y_{a-2} \quad x_{a-1} \quad y_{a-1} \quad x_{2a-3} \quad y_{2a-3} \quad x_{2a-2} \\
&x_a \quad y_0 \quad x_{a+1} \quad y_1 \quad x_{2a-2} \quad y_{a-2} \quad x_0 \quad y_{2a-4} \quad x_{a-2} \quad y_{2a-3}
\end{align*}
\]

Figure 12. \( \mathbb{C}_0 \cup \mathbb{C}_1 \).

\[
\begin{align*}
&x_0 \quad y_0 \quad x_1 \quad y_1 \quad x_{a-2} \quad y_{a-2} \quad x_{a-1} \quad y_{a-1} \quad x_{2a-3} \quad y_{2a-3} \quad x_{2a-2} \\
&x_a \quad y_0 \quad x_{a+1} \quad y_1 \quad x_{2a-2} \quad y_{a-2} \quad x_0 \quad y_{2a-4} \quad x_{a-2} \quad y_{2a-3}
\end{align*}
\]

Figure 13. \( \mathbb{P}_{00} \cup \mathbb{P}_{01} \).

By applying a procedure similar to Case 1, the remaining pairs of cycles \( \mathbb{C}_{0\mu} \) and \( \mathbb{C}_{0(\mu+1)} \), \( 2 \leq \mu \neq a \leq r - 1 \) decomposes into pairs of paths. Hence the graph \( K_{m,n}(2) \) has the desired decomposition.

\[\blacksquare\]

**Theorem 7.** Let \( p, q \) be nonnegative integers and \( \{k, m, n, \lambda\} \in \mathbb{N} \) such that \( k \equiv \lambda \equiv 0 \) (mod 2), \( m+n > k \geq 4 \), and \( k \) divides \( 2mn \). If \( m, n \geq k/2 \), \( k(p+q) = \lambda mn \), and \( p \neq 1 \), then the graph \( K_{m,n}(\lambda) \) has a \((k;p,q)\)-decomposition.
Theorem 9. Let \( \lambda \geq 3 \). In this section, we investigate the existence of a \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\) when \(\lambda \equiv k \equiv 0 \pmod{4}\). By Lemma 6 and Remark 5, the graph \((\lambda/2)K_{m,n}(2)\) has a \((k; p, q)\)-decomposition. Hence the graph \(K_{m,n}(\lambda)\) has the desired decomposition.

Remark 8.

1. Let \(k, m, n\) be positive even integers such that \(k \geq 4\). If the graph \(K_{m,n}(\lambda)\) has a \((k; p, q)\)-decomposition, then for every positive integer \(x\), the graph \(K_{m,n}(x\lambda)\) has a \((k; p, q)\)-decomposition.

2. Let \(k, m, n\) be positive even integers such that \(k \geq 4\). If the graph \(K_{m,n}(\lambda)\) has a \((k; p, q)\)-decomposition, then for all positive integers \(r\) and \(s\), the graph \(K_{rm,sn}(\lambda)\) has a \((k; p, q)\)-decomposition.

3. Let \(k, n_1, n_2, \ldots, n_m\) be positive even integers such that \(k \geq 4\). If the graph \(K_{n_i,n_j}(\lambda)\), for \(1 \leq i \neq j \leq m\) has a \((k; p, q)\)-decomposition, then the graph \(K_{n_1, n_2, \ldots, n_m}(\lambda)\) has a \((k; p, q)\)-decomposition.

3. \((k; p, q)\)-Decomposition of \(K_{m,n}(\lambda)\), When \(\lambda \geq 3\)

In this section, we investigate the existence of a \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(\lambda \geq 3\) and \(\lambda m \equiv \lambda n \equiv k \equiv 0 \pmod{2}\).

Theorem 9. Let \(\{k, m, n, \lambda\} \in \mathbb{N}\) and \(i, j\) be nonnegative integers such that \(\lambda \geq 3\), \(\lambda m \equiv \lambda n \equiv 0 \pmod{2}\), and \(k \equiv 0 \pmod{4}\). If \(K_{\frac{k}{2} + i, \frac{k}{2} + j}(\lambda)\), \(0 \leq i, j \leq k/2\) has a \((k; p, q)\)-decomposition, then the graph \(K_{m,n}(\lambda)\), where \(m, n \geq k\), has a \((k; p, q)\)-decomposition.

Proof. By the hypothesis, let \(m = tk + x\) and \(n = sk + y\), where \(t\) and \(s\) are positive integers, \(x\) and \(y\) are nonnegative integers such that \(0 \leq x, y < k\).

When \(x = y = 0\), we can write \(K_{m,n}(\lambda) = K_{tk,sk}(\lambda) = \lambda tsK_{k,k}\). When \(x = y = k/2\), we can write

\[
K_{m,n}(\lambda) = K_{(t-1)k+\frac{k}{2},(s-1)k+\frac{k}{2}}(\lambda)
\]

\[
= K_{(t-1)k,(s-1)k}(\lambda) + K_{(t-1)k,\frac{k}{2}}(\lambda) + K_{\frac{k}{2},(s-1)k}(\lambda) + K_{\frac{k}{2},\frac{k}{2}}(\lambda)
\]

\[
= ((t-1)(s-1)\lambda)K_{k,k} + (t-1)\lambda K_{k,\frac{k}{2}} + (s-1)\lambda K_{\frac{k}{2},k} + \lambda K_{\frac{k}{2},\frac{k}{2}}.
\]

Since \(k \equiv 0 \pmod{4}\), by Theorem 1 the graphs \(K_{k,k}, K_{k,\frac{k}{2}}, K_{\frac{k}{2},k}\) and \(K_{\frac{k}{2},\frac{k}{2}}\) have a \((k; p, q)\)-decomposition. Hence the graph \(K_{m,n}(\lambda)\) has the desired decomposition.
Case 1: \( x = 0 \) and \( 0 < y < k \). When \( 0 < y < k/2 \), we can write

\[
K_{m,n}(\lambda) = K_{tk,(s-1)k+\frac{t}{2}+\frac{s}{2}+y}(\lambda) = K_{tk,(s-1)k+\frac{t}{2}}(\lambda) \oplus K_{tk,y+\frac{s}{2}}(\lambda) \\
= (t\lambda)K_{k,(s-1)k+\frac{t}{2}} \oplus tK_{k,y+\frac{s}{2}}(\lambda) \\
= (t(s-1)\lambda)K_{k,k} \oplus (t\lambda)K_{k,\frac{t}{2}} \oplus tK_{k,y+\frac{s}{2}}(\lambda).
\]

By Theorem 1, the graphs \( K_{k,k} \), \( K_{k,\frac{t}{2}} \), both have a \((k; p, q)\)-decomposition and by the hypothesis, the graph \( K_{k,y+\frac{s}{2}}(\lambda) \) has a \((k; p, q)\)-decomposition.

When \( k/2 \leq y < k \), we can write

\[
K_{m,n}(\lambda) = K_{tk,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda) \\
= (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda).
\]

By Theorem 1, the graph \( K_{k,k} \) has a \((k; p, q)\)-decomposition and by the hypothesis, the graph \( K_{k,y}(\lambda) \) has a \((k; p, q)\)-decomposition. Hence the graph \( K_{m,n}(\lambda) \) has the desired decomposition.

Case 2: \( k/2 < x < k \) and \( k/2 \leq y < k \). We can write

\[
K_{m,n}(\lambda) = K_{tk+x,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda) \oplus K_{x,sk}(\lambda) \oplus K_{x,y}(\lambda) \\
= (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda) \oplus sK_{x,k}(\lambda) \oplus K_{x,y}(\lambda),
\]

and \( \lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda(tsk+sx+ty)+\lambda xy/k \). By Theorem 1, the graph \( K_{k,k} \) has a \((k; p, q)\)-decomposition and by the hypothesis, the graphs \( K_{k,y}(\lambda) \) and \( K_{x,k}(\lambda) \) both have a \((k; p, q)\)-decomposition. Since \( k \) divides \( \lambda mn \), we have \( k \) divides \( \lambda xy \) and also \( k/2 \leq x, y < k \), then by the hypothesis, \( K_{x,y}(\lambda) \) has a \((k; p, q)\)-decomposition. Hence, by Remark 5, the graph \( K_{m,n}(\lambda) \) has the desired decomposition.

Case 3: \( 0 < x, y \leq k/2 \). We can write

\[
K_{m,n}(\lambda) = K_{(t-1)k+(k+x),(s-1)k+(k+y)}(\lambda) \\
= K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,sk+y}(\lambda) \oplus K_{k,x+(s-1)k}(\lambda) \oplus K_{k,x,k+y}(\lambda) \\
= (t-1)(s-1)K_{k,k}(\lambda) \oplus (t-1)K_{k,k}(\lambda) \oplus (s-1)K_{k,x,k+y}(\lambda) \\
\oplus K_{k/2,k+y}(\lambda) \oplus K_{k/2,x,k+y}(\lambda) \\
= \lambda(t-1)(s-1)K_{k,k} \oplus (t-1)K_{k,k/2}(\lambda) \oplus (t-1)K_{k,k/2}(\lambda) \\
\oplus (s-1)K_{k/2,k}(\lambda) \oplus (s-1)K_{k/2,k}(\lambda) \oplus K_{k/2,k+y}(\lambda) \\
\oplus K_{k/2,x,k+2}(\lambda) \oplus K_{k/2,x,k/2+y}(\lambda),
\]

and \( \lambda mn/k = \lambda(tk+x)(sk+y)/k = \lambda k(t-1)(s-1) + \lambda(t-1)(k+y) + \lambda(k+x)(s-1) + \lambda(k+x+y) + (\lambda xy)/k \). By Theorem 1, the graphs \( K_{k,k} \) and \( K_{k/2,k} \) both
have a \((k; p, q)\)-decomposition and by the hypothesis, the graphs \(K_{k, k/2 + y}(\lambda)\), \(K_{k/2 + x, k}(\lambda)\), both have a \((k; p, q)\)-decomposition. Since \(k\) divides \(\lambda mn\) and \(k \equiv 0 \pmod{4}\), we have \(k\) divides \(\lambda(k/2 + x)(k/2 + y)\), \(2\) divides \(\lambda x\), and \(2\) divides \(\lambda y\) and \(k/2 \leq (k/2 + x), (k/2 + y) \leq k\). Then by the hypothesis, the graphs \(K_{k/2 + x, k/2 + y}(\lambda)\), \(K_{k/2, k}(\lambda)\), and \(K_{k/2, k/2 + y}(\lambda)\) have a \((k; p, q)\)-decomposition. The graph \(K_{k/2, k/2 + y}(\lambda)\) can be viewed as \(K_{k/2, k/2}(\lambda) \oplus K_{k/2, k/2 + y}(\lambda) = \lambda K_{k/2, k/2} \oplus K_{k/2, k/2 + y}(\lambda)\). By Theorem 2, the graph \(K_{k/2, k/2}\) has a \(C_k\)-decomposition and by the hypothesis, the graph \(K_{k/2, k/2 + y}(\lambda)\) has a \((k; p, q)\)-decomposition. Now for any pair of cycles of length \(k\), one from the graph \(\lambda K_{k/2, k/2}\), say \(C_\alpha\) and the other from the graph \(K_{k/2, k/2 + y}(\lambda)\), say \(C_\beta\), we have a common vertex in \(C_\alpha \oplus C_\beta\), say \(v\). Then by the Construction 4 we have two edge-disjoint paths of length \(k\) from \(C_\alpha\) and \(C_\beta\). By applying a similar procedure to the remaining pairs of cycles, we have edge-disjoint pairs of paths. Hence the graph \(K_{k/2, k + y}(\lambda)\) has a \((k; p, q)\)-decomposition. Therefore, by Remark 5, the graph \(K_{m, n}(\lambda)\) has the desired decomposition.

**Case 4:** \(0 < x \leq k/2\) and \(k/2 < y < k\). We can write

\[
K_{m, n}(\lambda) = K_{(t-1)k + (k + x), sk + y}(\lambda) = K_{(t-1)k, sk}(\lambda) \oplus K_{(t-1)k + (k + x), sk + y}(\lambda) = ((t - 1)s\lambda)K_{k, k} + (t - 1)K_{k, y}(\lambda) + sK_{k + x, k}(\lambda) + K_{k, k + y}(\lambda) + (t - 1)s\lambda)K_{k, k} \oplus (t - 1)K_{k, y}(\lambda) + sK_{k + x, k}(\lambda) + K_{k + y, y}(\lambda),
\]

and \(\lambda mn/k = \lambda(tk + x)(sk + y)/k = \lambda((t-1)sk + (t-1)y + sk/2 + s(k/2 + x)) + \lambda(k + x)y/k\). By Theorem 1, the graphs \(K_{k, k}\) and \(K_{k/2, k}\) both have a \((k; p, q)\)-decomposition. Since \(k\) divides \(\lambda mn\), we have 2 divides \(\lambda y\), \(k\) divides \(x\) and also \(k/2 \leq (k/2 + x), y \leq k\), then by the hypothesis, the graphs \(K_{k, y}(\lambda)\), \(K_{k/2, x, k}(\lambda)\), and \(K_{k/2, x, y}(\lambda)\) have a \((k; p, q)\)-decomposition. Hence, by Remark 5, the graph \(K_{m, n}(\lambda)\) has the desired decomposition.

**Theorem 10.** Let \(\{k, m, n, \lambda\} \in \mathbb{N}\) and \(i, j\) be nonnegative integers such that \(\lambda \geq 3\), \(\lambda m \equiv \lambda n \equiv 0 \pmod{2}\), and \(k \equiv 2 \pmod{4}\). If \(K_{k, k}^{i} \oplus k_{k/2, k}^{j}(\lambda)\), \(0 \leq i, j \leq k\) has a \((k; p, q)\)-decomposition, then the graph \(K_{m, n}(\lambda)\), where \(m, n \geq 3k/2\), has a \((k; p, q)\)-decomposition.

**Proof.** By the hypothesis, let \(m = tk + x\) and \(n = sk + y\), where \(t\) and \(s\) are positive integers, \(x\) and \(y\) are nonnegative integers such that \(0 \leq x, y < k\).
When $x = y = k/2$, we can write

$$K_{m,n}(\lambda) = K_{(t-1)k+(s-1)k+x, y}(\lambda)$$

$$= K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,\frac{3n}{2},(s-1)k}(\lambda) \oplus K_{\frac{3n}{2},(s-1)k}(\lambda)$$

$$= ((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)\lambda K_{\frac{3n}{2},k} \oplus (s-1)\lambda K_{\frac{3n}{2},\frac{3n}{2}}.$$

By Theorem 1, the graph $K_{k,k}$ has a $(k; p, q)$-decomposition and by the hypothesis, the graphs $K_{\frac{3n}{2},k}$ and $K_{\frac{3n}{2},\frac{3n}{2}}$ both have a $(k; p, q)$-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

**Case 1:** $0 \leq x, y < k/2$. When $0 \leq x, y < k/2$, we have $t, s \geq 2$. We can write

$$K_{m,n}(\lambda) = K_{(t-1)k+(k+x),(s-1)k+(k+y)}(\lambda)$$

$$= K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{t-1,k,k+y}(\lambda) \oplus K_{k+x,(s-1)k}(\lambda) \oplus K_{k,k+y}(\lambda)$$

$$= ((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)\lambda K_{k,k+y}(\lambda)$$

and $\lambda mn/k = \lambda(tk + x)(sk + y)/k = \lambda((t-1)(s-1)k + (s-1)(k + x) + (t-1)(k + y)) + \lambda(k + x)(k + y)/k$.

By Theorem 1, the graph $K_{k,k}$ has a $(k; p, q)$-decomposition and by the hypothesis, the graphs $K_{k,k+y}(\lambda)$ and $K_{k+x,k}(\lambda)$ both have a $(k; p, q)$-decomposition. Since $k$ divides $\lambda mn$, we have $k$ divides $\lambda(k + x)(k + y)$ and also $k/2 \leq (k + x), (k + y) \leq 3k/2$. Then by the hypothesis, the graph $K_{k,k+y}(\lambda)$ has a $(k; p, q)$-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

**Case 2:** $k/2 \leq x < k$ and $k/2 < y < k$. We can write $K_{m,n}(\lambda) = K_{tk+x,y}(\lambda)$

$$= K_{tk,x}K_{x,y}(\lambda) \oplus K_{tk,y}(\lambda) \oplus K_{tk,sk}(\lambda) \oplus K_{x,y}(\lambda) \oplus (ts)K_{k,k+y}(\lambda) \oplus sk_{k,y}(\lambda) \oplus sK_{x,y}(\lambda) \oplus K_{x,y}(\lambda),$$

and $\lambda mn/k = \lambda(tk + x)(sk + y)/k = \lambda(tk + sk + sy + ty)/k$. By Theorem 1, the graph $K_{k,k}$ has a $(k; p, q)$-decomposition and by the hypothesis, the graphs $K_{k,y}(\lambda)$ and $K_{x,y}(\lambda)$ both have a $(k; p, q)$-decomposition. Since $k$ divides $\lambda mn$, we have $k$ divides $\lambda xy$ and also $k/2 \leq x, y < k$, then by the hypothesis, the graph $K_{x,y}(\lambda)$ has a $(k; p, q)$-decomposition. Hence, by Remark 5, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

**Case 3:** $0 \leq x < k/2$ and $k/2 \leq y < k$. When $0 \leq x < k/2$ and $k/2 \leq y < k$, we have $t \geq 2$ and $s \geq 1$. We can write

$$K_{m,n}(\lambda) = K_{(t-1)k+(k+x),sk+y}(\lambda)$$

$$= K_{(t-1)k,(k+x),sk+y}(\lambda) \oplus K_{(t-1)k,y}(\lambda) \oplus K_{k+x,sk}(\lambda) \oplus K_{k+x,y}(\lambda)$$

$$= ((t-1)s\lambda)K_{k,k} \oplus (t-1)\lambda K_{k,y}(\lambda) \oplus sK_{k+x,k}(\lambda) \oplus K_{k+x,y}(\lambda),$$
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and \( \lambda mn/k = \lambda(tk + x)(sk + y)/k = \lambda((t - 1)sk + s(k + x) + (t - 1)y) + \lambda(k + x)y/k \). By Theorem 1, the graph \( K_{k,k} \) has a \((k; p, q)\)-decomposition and by the hypothesis, the graphs \( K_{k,y}(\lambda) \) and \( K_{k+x,k}(\lambda) \) both have a \((k; p, q)\)-decomposition. Since \( k \) divides \( \lambda mn \), we have \( k \) divides \( \lambda(k + x)y \) and also \( k/2 \leq (k + x), y \leq 3k/2 \), then by the hypothesis, the graph \( K_{k+x,y}(\lambda) \) has a \((k; p, q)\)-decomposition. Hence, by Remark 5, the graph \( K_{m,n}(\lambda) \) has the desired decomposition.

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