k-KERNELS AND SOME OPERATIONS IN DIGRAPHS

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Abstract

Let $D$ be a digraph. $V(D)$ denotes the set of vertices of $D$; a set $N \subseteq V(D)$ is said to be a $k$-kernel of $D$ if it satisfies the following two conditions: for every pair of different vertices $u, v \in N$ it holds that every directed path between them has length at least $k$ and for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an $xy$-directed path of length at most $k - 1$.

In this paper, we consider some operations on digraphs and prove the existence of $k$-kernels in digraphs formed by these operations from another digraphs.

Keywords: $k$-kernel, $k$-subdivision digraph, $k$-middle digraph and $k$-total digraph.

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1. Introduction

We refer the reader to [1] for general concepts. In this paper, $D$ denotes a digraph; $V(D)$ is the set of vertices and $A(D)$ denotes the set of arcs.

A directed path is a sequence $P = (x_0, x_1, \ldots, x_n)$ of distinct vertices of $D$ such that $(x_i, x_{i+1}) \in A(D)$ for each $i, 0 \leq i \leq n - 1$. The length of $P$ is $n$ and we denote $\ell(P) = n$. For $x, y \in V(D)$, the distance from $x$ to $y$ in $D$ is denoted as $d_D(x, y)$ and defined as: $d_D(x, y) = \min\{\ell(P) | P$ is an $xy$-directed path $\}$ whenever there exists an $xy$-directed path in $D$, otherwise, we define $d_D(x, y) = \infty$. If $P$ is a directed path and $a, b \in V(P)$, then $(a, P, b)$ denotes the $ab$-directed path contained in $P$.

A set $N \subseteq V(D)$ is said to be $k$-independent whenever for any two different vertices $x, y \in N$ we have $d_D(x, y) \geq k$ and $d_D(y, x) \geq k$. $N$ is said to be $(k - 1)$-absorbent whenever for each $x \in V(D) - N$ there exists $y \in N$ such that $d_D(x, y) \leq k - 1$. The set $N$ is said to be a $k$-kernel if it is $k$-independent and $(k - 1)$-absorbent.

We note that a 2-kernel is a kernel of a digraph in the sense of J. von Neumann and O. Morgenstern [20]. The problem of the existence of a kernel in a digraph has been studied in [2, 3, 4, 7, 17, 18].

The existence of kernels of digraphs formed by some operations from another digraphs have been studied by several authors, namely: M. Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel [16]; M. Harminc and T. Olejníková [11]; J. Topp [19], H. Galeana-Sánchez and V. Neumann-Lara [7, 8].

The concept of $k$-kernel was introduced by M. Kwaśnik in [14]. Clearly, this concept generalizes the concept of a kernel of a digraph. It has been studied by several authors: M. Harminc [9], M. Kwaśnik [14, 15], M. Kucharska [12, 13], H. Galeana-Sánchez [5, 6], A. Włoch and I. Włoch [21].

In [10], M. Harminc constructed all kernels of the line digraph of $D$ from the kernels of $D$ and in [19] the author considered some special digraphs: $S(D)$; $Q(D)$, $T(D)$ and $L(D)$ which were called the subdivision digraph, the middle digraph, the total digraph and the line digraph of $D$, respectively and studied some necessary or sufficient conditions for the existence or uniqueness of kernels of these digraphs.

In this paper, for a given digraph $D$ and any $k \geq 2$ we define: the $k$-subdivision $S^k(D)$, a generalization of the subdivision $S(D)$, the digraph $R^k(D)$, the $k$-middle digraph $Q^k(D)$ and the $k$-total digraph $T^k(D)$. Also the following results are proved: for any digraph $D$ and for any $k \geq 2$ the
digraphs $S^k(D)$, $R^k(D)$ and $Q^k(D)$ have a $k$-kernel. For any digraph $D$ and for $k \geq 3$ the digraph $T^k(D)$ has a $k$-kernel.

2. $k$-Kernels in: $S^k(D)$, $R^k(D)$, $Q^k(D)$ and $T^k(D)$

Let $D$ be a digraph. The line digraph $L(D)$ of $D$ is the digraph defined as follows: $V(L(D)) = A(D)$ and $(a = (u, v), b = (z, w)) \in A(L(D))$ if and only if $v = z$ [1].

[19]: For a given digraph $D$, the subdivision digraph $S(D)$ of $D$ is defined by: $V(S(D)) = V(D) \cup A(D)$ and

$$\Gamma^+(x) = \begin{cases} \{x\} \times \Gamma^+_D(x), & \text{whenever } x \in V(D), \\ \{v\}, & \text{whenever } x = (u, v) \in A(D). \end{cases}$$

Notice that for a vertex $x$ of the subdivision digraph of $D$ we have the following: If $x$ corresponds to a vertex of $D$, then $x$ is adjacent to the arcs which are incident from $x$ in $D$; and if $x$ corresponds to an arc of $D$, then $x$ is adjacent only to the terminal endpoint of $x$. Also notice that $S(D)$ is obtained from $D$ by changing each arc of $D$ for a directed path of length two.

Let $D$ be a digraph. We define the $k$-subdivision digraph of $D$, denoted $S^k(D)$, as follows:

$$S^k(D) = S(D) - \{(u, a) | a \in A(D) \text{ and } u \text{ is the initial endpoint of } a\} \cup \bigcup_{a \in A(D)} \beta_a$$

for each $a = (u, v) \in A(D)$, $\beta_a = (a_0 = u, a_1, \ldots, a_{n(a)k+k-1} = a = (u, v))$ is a $ua$-directed path whose length is $\equiv k - 1 (\bmod k)$ $(n(a) \in \mathbb{N})$ and the following two properties hold:

(i) $V(\beta_a) \cap V(S(D)) = \{u, a\},$

(ii) For any $a, b \in A(D)$ with $a \neq b$ we have $(V(\beta_a) - \{u\}) \cap V(\beta_b) = \emptyset.$

Notice that $S^k(D)$ is obtained from $D$ by substituting each arc of $D$ for a directed path whose length is $\equiv 0 (\bmod k)$ (for an example see Figure 1).

We write $V^0(D) = \{x \in V(D) | \delta^+_D(x) = 0\}$. 
Finally, we define the digraphs $R^k(D)$, $Q^k(D)$ and $T^k(D)$ as follows $R^k(D) = S^k(D) \cup D$, $Q^k(D) = S^k(D) \cup L(D)$ and $T^k(D) = S^k(D) \cup D \cup L(D)$ (for an example see Figure 2).
Theorem 2.1. For any digraph $D$ and for any integer $k$ $(k \geq 2)$, the $k$-subdivision digraph $S^k(D)$ of $D$ has a $k$-kernel.

**Proof.** Let $D$ and $S^k(D)$ be digraphs as in the hypothesis. For each $a \in A(D)$ we denote $\mathcal{N}_a = \{a_i \in V(\beta_a) \mid i \equiv 0 \pmod{k}\}$. We will prove that $\mathcal{N} = V_0(D) \cup \bigcup_{a \in A(D)} \mathcal{N}_a$ is a $k$-kernel of $S^k(D)$. Observe that $V(D) \subseteq \mathcal{N}$.

**Claim 1.** $\mathcal{N}$ is a $k$-independent set of vertices of $S^k(D)$.

Let $x, y \in \mathcal{N}$, $x \neq y$. We will prove $d_{S^k(D)}(x, y) \geq k$ and $d_{S^k(D)}(y, x) \geq k$.

**Case 1.** $x \in V_0(D)$ and $y \in V_0(D)$.
Since $\delta_{S^k(D)}^+(x) = \delta_{S^k(D)}^-(x) = 0$ and $\delta_{S^k(D)}^+(y) = \delta_{S^k(D)}^-(y) = 0$, it follows that $d_{S^k(D)}(x, y) = d_{S^k(D)}(y, x) = \infty$.

**Case 2.** $x \in V_0(D)$ and $y \in \bigcup_{a \in A(D)} \mathcal{N}_a$.
Since $\delta_{S^k(D)}^+(x) = \delta_{S^k(D)}^-(x) = 0$, we have $d_{S^k(D)}(x, y) = \infty$. Let $c = (u, v) \in A(D)$ such that $y \in \mathcal{N}_c$. From the definition of $S^k(D)$ we have $d_{S^k(D)}(y, x) = d_{S^k(D)}(y, c = (u, v)) + d_{S^k(D)}(c, x)$. Now since $y = c_i$ with $i \equiv 0 \pmod{k}$ and $\ell(\beta_c) \equiv k - 1 \pmod{k}$ it follows that $d_{S^k(D)}(y, c = (u, v)) = d_{S_{\beta_c}}(y, c) \geq k - 1$. Clearly, $d_{S^k(D)}(c, x) \geq 1$ (as $c \in A(D)$ and $x \in V_0(D) \subseteq V(D)$). Therefore $d_{S^k(D)}(y, x) \geq (k - 1) + 1 = k$.

**Case 3.** $x \in \bigcup_{a \in A(D)} \mathcal{N}_a$ and $y \in V_0(D)$.
Proceed exactly as in Case 2 interchanging $x$ with $y$.

**Case 4.** $x \in \bigcup_{a \in A(D)} \mathcal{N}_a$ and $y \in \bigcup_{a \in A(D)} \mathcal{N}_a$.

**Case 4.1.** There exists $c = (u, v) \in A(D)$ such that $\{x, y\} \subseteq \mathcal{N}_c$.
From the definition of $\mathcal{N}_c$ we have $x = c_{mk}$ and $y = c_{tk}$ for some $0 \leq m \leq n(c)$, $0 \leq t \leq n(c)$. Assume without loss of generality $t > m$.

From the definition of $S^k(D)$ and the fact $x \neq v$ (as $\ell(\beta_c) \equiv k - 1 \pmod{k}$) we have: $d_{S^k(D)}(x, y) = d_{S_{\beta_c}}(x, y) = (t - m)k \geq k$. On the other hand, we have $d_{S^k(D)}(y, c) = d_{S_{\beta_c}}(y, c) \geq k - 1$ and $d_{S^k(D)}(c, v) = 1$, we obtain $d_{S^k(D)}(y, x) \geq k$.

**Observation 1.** Observe that in this case we have the same inequalities when we are working in $Q^k(D)$, i.e., $d_{Q^k(D)}(y, x) \geq k$, because the definition of $Q^k(D)$ implies: $d_{Q^k(D)}(y, x) = d_{Q_{\beta_c}}(y, x) + d_{Q^k(D)}(c, v) + d_{Q^k(D)}(v, x)$. And clearly, $d_{Q^k(D)}(y, c) \geq k - 1$ and $d_{Q^k(D)}(c, x) \geq 1$. 
Case 4.2. $x \in \mathcal{M}_a$ and $y \in \mathcal{M}_b$ for some $a, b \in A(D)$ with $a \neq b$. Assume without loss of generality that $a = (u, v)$ and $b = (w, z)$.

$d_{S^k(D)}(x, y) = d_{S^k(D)}(x, a) + d_{S^k(D)}(a, v) + d_{S^k(D)}(v, y)$. From the definition of $\mathcal{M}_a$ we have $d_{S^k(D)}(x, a) \geq k - 1$ and from the definition of $S^k(D)$, $d_{S^k(D)}(a, v) = 1$. Therefore $d_{S^k(D)}(x, y) \geq k$.

Observation 2. Notice that in this case we have $d_{Q^k(D)}(x, a) = d_{S^k(D)}(x, a)$ and $d_{Q^k(D)}(a, y) \geq 1$ (as $a \neq y$). Thus $d_{Q^k(D)}(x, y) = d_{Q^k(D)}(x, a) + d_{Q^k(D)}(a, y) \geq k$.

Interchanging $x$ with $y$ we obtain $d_{S^k(D)}(y, x) \geq k$.

Claim 2. \( \mathcal{M} \) is a \((k - 1)\)-absorbent set of vertices of $S^k(D)$.

Let $x \in V(S^k(D) - \mathcal{M})$. We will prove that there exists $y \in \mathcal{M}$ such that $d_{S^k(D)}(x, y) \leq k - 1$. Since $V^0 \subseteq \mathcal{M}$, it follows from the definition of $S^k(D)$ and the fact $x \in V(S^k(D) - \mathcal{M})$ that $x \in \bigcup_{a \in A(D)} \beta_a$. Let $c = (u, v) \in A(D)$ be such that $x \in \beta_c$.

Case 1. $x \in \beta_c - \{c_i | n(c)k + 1 \leq i \leq n(c)k + (k - 1)\}$.

Since $x \notin \mathcal{M}$ (and then $x \notin \mathcal{M}_c$), it follows that $x = c_{mk+j}$ for some $m$ and $j$ with $0 \leq m \leq n(c)$ and $1 \leq j \leq k - 1$. From the definition of $S^k(D)$ we have $d_{S^k(D)}(x, c_{(m+1)k}) = d_{\beta_c}(c_{mk+j}, c_{(m+1)k}) = k - j \leq k - 1$. Clearly, $c_{(m+1)k} \in \mathcal{M}$.

Case 2. $x \in \{c_i | n(c)k + 1 \leq i \leq n(c)k + (k - 1) = (u, v) = c\}$.

Clearly, $d_{S^k(D)}(x, v) = d_{S^k(D)}(x, c) + d_{S^k(D)}(c, v); d_{S^k(D)}(x, c) \leq k - 2$ and $d_{S^k(D)}(c, v) = 1$. Thus $d_{S^k(D)}(x, v) \leq k - 1$ with $v \in \mathcal{M}$ (recall $V(D) \subseteq \mathcal{M}$).

Theorem 2.2. For any digraph $D$ and for any integer $k$ ($k \geq 2$), the $k$-middle digraph $Q^k(D)$ of $D$ has a $k$-kernel.

Proof. Consider the set $\mathcal{M} \subseteq V(S^k(D)) = V(Q^k(D))$ defined in the proof of Theorem 2.1. Since $S^k(D)$ is a spanning subdigraph of $Q^k(D)$ and $\mathcal{M}$ is a $(k - 1)$-absorbent set of vertices of $S^k(D)$, it follows that $\mathcal{M}$ is a $(k - 1)$-absorbent set of vertices of $Q^k(D)$.

The proof that $\mathcal{M}$ is $k$-independent in $Q^k(D)$ is the same as the proof that $\mathcal{M}$ is $k$-independent in $S^k(D)$, we only need to recall Observations 1 and 2 given along this proof.
Theorem 2.3. Let $D$ be any digraph and for any integer $k$ ($k \geq 2$), then the digraph $R^k(D)$ has a $k$-kernel.

Proof. Let $D$, $k$ and $R^k(D)$ be as in the hypothesis. For each $a = (u, v) \in A(D)$ we define $\mathcal{N}_a$ as follows: $\mathcal{N}_a$ is the unique $k$-kernel of $(\beta_a - \{u\}) \cup\{(a = (u, v), v)\}$ whenever $\delta^+_{\mathcal{D}}(v) = 0$. And $\mathcal{N}_a = \{a_i \in V(\beta_a) | i \equiv 1 \pmod{k}\}$ whenever $\delta^+_{\mathcal{D}}(v) > 0$. We write $B^0 = \{x \in V(D) | \delta^+_{\mathcal{D}}(x) = \delta^+_{\mathcal{D}}(x) = 0\}$. We will prove that $\mathcal{R} = \bigcup_{a \in A(D)} \mathcal{N}_a \cup B^0$ is a $k$-kernel of $R^k(D)$. First, observe that $V^0(D) \subseteq \mathcal{N}$.

Claim 3. $\mathcal{R}$ is a $k$-independent set of $R^k(D)$.

Let $x, y \in \mathcal{R}$ with $x \neq y$. We will prove that $d_{R^k(D)}(x, y) \geq k$ and $d_{R^k(D)}(y, x) \geq k$. Observe that if $x \in B^0$, then $d_{R^k(D)}(x, y) = d_{R^k(D)}(y, x) = \infty \forall y \in V(R^k(D))$.

Case 1. There exists $c = (u, v) \in A(D)$ such that $\{x, y\} \subseteq \mathcal{N}_c$.

Case 1.1. $\delta^+_{\mathcal{D}}(v) = 0$. In this case, we have $\mathcal{N}_c = \{c_i \in V(\beta_c) | i \equiv 0 \pmod{k}, i > 0\} \cup \{v\}$.

We assume without loss of generality that $x = c_{mk}$ with $1 \leq m \leq n(c)$ and, $y = c_{tk}$ with $m < t$ or $y = v$.

When $y = c_{tk}$, we have $d_{R^k(D)}(x, y) = (t - m)k \geq k$. When $y = v$, we have $d_{R^k(D)}(x, v) = d_{R^k(D)}(x, c) + d_{R^k(D)}(c, v)$. Since $d_{R^k(D)}(x, c) \geq k - 1$ and $d_{R^k(D)}(c, v) = 1$, we conclude $d_{R^k(D)}(x, v) = y \geq k$.

Now, from the definition of $R^k(D)$ we have: $d_{R^k(D)}(y, x) = d_{R^k(D)}(y, v) + d_{R^k(D)}(v, x)$. Since $\delta^+_{\mathcal{D}}(v) = 0$ we have $d_{R^k(D)}(v, x) = \infty$. Thus $d_{R^k(D)}(y, x) \geq k$.

Case 1.2. $\delta^+_{\mathcal{D}}(v) > 0$. In this case we have $\mathcal{N}_c = \{c_i \in V(\beta_c) | i \equiv 1 \pmod{k}\}$. We assume without loss of generality that $x = c_{mk+1}$, $y = c_{tk+1}$ with $0 \leq m < t$. Clearly, $d_{R^k(D)}(x, y) = (t - m)k \geq k$ and $d_{R^k(D)}(y, x) = d_{R^k(D)}(y, c) + d_{R^k(D)}(c, v) + d_{R^k(D)}(v, x)$. From the definition of $R^k(D)$ we have $d_{R^k(D)}(y, c) \geq k - 2$, $d_{R^k(D)}(c, v) = 1$ and $d_{R^k(D)}(v, x) \geq 1$ (because $v \neq x$, be as $m < t$). Thus $d_{R^k(D)}(y, x) \geq k$.

Case 2. $x \in \mathcal{N}_b$ and $y \in \mathcal{N}_c$ with $b = (u, v)$, $c = (w, z)$, $b \neq c$.

From the definition of $R^k(D)$ we have $d_{R^k(D)}(x, y) = d_{R^k(D)}(x, b = (u, v)) + d_{R^k(D)}(b, v) + d_{R^k(D)}(v, w) + d_{R^k(D)}(w, y)$. When $\delta^+_{\mathcal{D}}(v) = 0$, we obtain $d_{R^k(D)}(v, w) = \infty$ and then $d_{R^k(D)}(x, y) \geq k$. 
When \( \delta^+_D(v) > 0 \), we obtain \( \mathcal{N}_b = \{ b_i \in V(\beta_b) | i \equiv 1(\text{mod } k) \} \) and \( d^k_D(x, b) \geq k - 2 \) also from the definition of \( R^k_D \), \( d^k_D(b, v) = 1 \). If \( v \neq w \), then \( d^k_D(v, w) \geq 1 \) and we conclude that \( d^k_D(x, y) \geq k \). If \( v = w \), then \( \delta^+_D(w) = 0 \), \( w \notin \mathcal{N}_b \), and \( w \neq y \); therefore \( d^k_D(w, y) \geq 1 \), and we conclude again that \( d^k_D(x, y) \geq k \).

Analogously, it can be proved \( d^k_D(y, x) \geq k \).

**Claim 4.** \( \mathcal{R} \) is a \((k - 1)\)-absorbent set of vertices of \( R^k_D \).

We will prove that for any \( z \in V(R^k(D) - \mathcal{R}) \) there exists \( w \in \mathcal{R} \) such that \( d^k_D(z, w) \leq k - 1 \).

Let \( z \in V(R^k(D) - \mathcal{R}) \). We have observed that \( V^0(D) \subseteq \mathcal{R} \). Thus \( z \in \bigcup_{a \in A(D)} V(\beta_a) \). Take \( c = (u, v) \in A(D) \) such that \( z \in V(\beta_c) \).

**Case 1.** \( \delta^+_D(v) = 0 \). In this case, \( \mathcal{N}_c = \{ c_i \in V(\beta_c) | i \equiv 0(\text{mod } k), i \geq 1 \} \cup \{ v \} \). Since \( z \notin \mathcal{N}_c \), then \( z = c_0 \) or \( z = c_{mk+j} \) with \( 1 \leq j \leq k - 1 \) and \( 0 \leq m \leq n(c) \).

If \( z = c_0 = u \), then from the definition of \( R^k_D \) we have \( (z = u, v) \in A(R^k(D)) \) and \( d^k_D(z, v) = 1 \leq k - 1 \) with \( v \in \mathcal{R} \). If \( z = c_{mk+j} \), then \( d^k_D(c_{mk+j}, c_{(m+1)k}) = k - j \leq k - 1 \) whenever \( m \neq n(c) \), and \( d^k_D(c_{mk+j}, c_{(m+1)k}) = k - j \leq k - 1 \) whenever \( m = n(c) \) (recall that \( z = c_{mk+j}, c = c_{n(k)+c(k-1)} \) and \( d^k_D(c = (u, v), v) = 1 \)).

**Case 2.** \( \delta^+_D(v) > 0 \). In this case, \( \mathcal{N}_c = \{ c_i \in V(\beta_c) | i \equiv 1(\text{mod } k) \} \).

When \( z \in V(\beta_c) - \{ c_i \in V(\beta_c) | k + 2 \leq i \leq n(c)k + (k - 1) \} \), we have two possibilities: If \( z = c_0 \), then \( d^k_D(z, c_1) = 1 \leq k - 1 \) with \( c_1 \in \mathcal{N} \) \( \subseteq \mathcal{R} \). If \( z \neq c_0 \), then \( z = c_{mk+j} \) with \( 2 \leq j \leq k, 0 \leq m < n(c) \) and \( d^k_D(z, c_{(m+1)k+1}) \leq k - 1 \) with \( c_{(m+1)k+1} \in \mathcal{R} \).

When \( z \in \{ c_i \in V(\beta_c) | k + 2 \leq i \leq n(c)k + (k - 1) \} \), we recall that \( \delta^+_D(v) > 0 \). Thus there exists \( b = (v, w) \in A(D) \). We consider \( \beta_b \). Consider two possibilities: If \( \delta^+_D(w) > 0 \), then \( \mathcal{N}_b = \{ b_i \in V(\beta_b) | i \equiv 1(\text{mod } k) \} \); and it follows that \( d^k_D(z, b_1) = d^k_D(z, c) + d^k_D(c, v) + d^k_D(v, b_1) \leq k - 3 + 1 + 1 = k - 1 \) with \( b_1 \in \mathcal{R} \). If \( \delta^+_D(w) = 0 \), then \( w \in \mathcal{R} \), and \( d^k_D(z, w) = d^k_D(z, c) + d^k_D(c, v) + d^k_D(v, w) \leq k - 3 + 1 + 1 = k - 1 \).

**Theorem 2.4.** For any digraph \( D \) and for any integer \( k \) \((k \geq 3)\), the digraph \( T^k(D) \) has a \( k \)-kernel.
**Proof.** Let $k$, $D$ and $T^k(D)$ be as in the hypothesis. For each $a = (u, v) \in A(D)$ we define $\mathfrak{N}_a$ as follows: If $\delta_D^+(v) = 0$, then $\mathfrak{N}_a$ is the $k$-kernel of $(\beta_a - \{u\}) \cup \{v, a = (u, v)\}$, i.e., $\mathfrak{N}_a = \{a_i|1 \leq i, i \equiv 0 \text{ (mod } k)\} \cup \{v\}$. If $\delta_D^+(v) > 0$, then $\mathfrak{N}_a = \{a_i|i \equiv 1 \text{ (mod } k)\}$. We write $B^0 = \{x \in V(D)|\delta_D^+(x) = \delta_D^-(x) = 0\}$. We will prove that $\mathfrak{N} = \bigcup_{a \in A(D)} \mathfrak{N}_a \cup B^0$ is a kernel of $T^k(D)$. Observe that $V^0(D) \subseteq \mathfrak{N}$.

**Observation 3.** Notice that since $k \geq 3$, we have $a_{n(a)k+1} \neq a = a_{n(a)k+(k-1)}$, therefore $a \not\in \mathfrak{N}$, for each $a \in A(D)$.

**Claim 5.** $\mathfrak{N}$ is a $k$-independent set of vertices of $T^k(D)$.

Let $x, y \in \mathfrak{N}$ with $x \neq y$. We will prove that $d_{T^k(D)}(x, y) \geq k$ and $d_{T^k(D)}(y, x) \geq k$. Observe that if $x \in B^0$, then $d_{T^k(D)}(x, y) = d_{T^k(D)}(y, x) = \infty$ for each $y \in V(T^k(D))$.

**Case 1.** There exists $c = (u, v) \in A(D)$ such that $\{x, y\} \subseteq \mathfrak{N}_c$.

**Case 1.1.** $\delta_D^+(v) = 0$. In this case, $\mathfrak{N}_c = \{c_i|1 \leq i, i \equiv 0 \text{ (mod } k)\} \cup \{v\}$. Clearly, we may assume $x = c_m$, $y = c_n$ with $1 \leq t \leq n(c)$ and $m < t$ or $y = v$.

If $y = c_k$, then $d_{T^k(D)}(x, y) = (t - m)k \geq k$. If $y = v$, then $d_{T^k(D)}(x, y) = d_{\beta_k}(x, c) + d_{T^k(D)}(c, v) \geq k - 1 + 1 = k$.

Now from the definition of $T^k(D)$, we have $d_{T^k(D)}(y, x) = d_{T^k(D)}(y, c) + d_{T^k(D)}(c, x)$.

If $y \neq v$, then $d_{T^k(D)}(y, c) = d_{\beta_k}(y, c) \geq k - 1$. From Observation 3 $c \neq x$, so $d_{T^k(D)}(c, x) \geq 1$ and we conclude that $d_{T^k(D)}(y, x) \geq k$.

If $y = v$, then $d_{T^k(D)}(y, x) = \infty$, as $\delta_D^+(v) = 0$.

**Case 1.2.** $\delta_D^+(v) > 0$. In this case, $\mathfrak{N}_c = \{c_i \in \beta_c|1 \equiv 1 \text{ (mod } k)\}$ and clearly, we may assume $x = c_{mk+1}$, $y = c_{mk+1}$ with $0 \leq m < t \leq n(c)$. Therefore $d_{T^k(D)}(x, y) = (t - m)k \geq k$. Now from the definition of $T^k(D)$ we have $d_{T^k(D)}(y, x) = d_{T^k(D)}(y, c) + d_{T^k(D)}(c, x)$. Clearly, $d_{T^k(D)}(y, c) \geq k - 2$.

Since $c \in A(D), c = (u, v)$ and $x \neq v$, we have $(c, x) \not\in A(T^k(D))$ (recall the definition of $T^k(D)$).

Hence $d_{T^k(D)}(c, x) \geq 2$. We conclude that $d_{T^k(D)}(y, x) \geq k$.

**Case 2.** $x \in \mathfrak{N}_b$ and $y \in \mathfrak{N}_c$ for $b = (u, v), c = (w, z)$ with $\{b, c\} \subseteq A(D), b \neq c$. From the definition of $T^k(D)$ we have $d_{T^k(D)}(x, y) = d_{T^k(D)}(x, b) + d_{T^k(D)}(b, y)$.
Case 2.1. $\delta_D^+(v) = 0$. In this case, $x = b_{mk}$ with $1 \leq m \leq n(b)$ or $x = v$. If $x = b_{mk}$, then $d_{T^k(D)}(x, b) \geq k - 1$; and from Observation 3 $b \neq y$ which implies $d_{T^k(D)}(b, y) \geq 1$. We conclude that $d_{T^k(D)}(x, y) = \infty$ (as $\delta_D^+(v) = \delta_{T^k(D)}^+(v) = 0$).

Case 2.2. $\delta_D^+(v) > 0$. In this case, $H_b = \{b_i \in V(\beta_b)|i \equiv 1(\text{mod } k)\}$. From the definition of $T^k(D)$ we have $d_{T^k(D)}(x, y) = d_{T^k(D)}(x, b) + d_{T^k(D)}(b, y)$. Clearly, $d_{T^k(D)}(x, b) \geq k - 2$. Since $b \not\in H$ (from Observation 3) and $y \in H$, then $y \neq b$. Moreover, $k \geq 3$ implies $n(b)k + 1 \neq n(b)k + (k - 1)$ and $y \neq v$. Finally, $d(b, y) = 1$ implies $y \in A(D)$ and by Observation 3 also $y \not\in H$, a contradiction. Therefore $d_{T^k(D)}(b, y) \geq 2$. We conclude that $d_{T^k(D)}(x, y) \geq k$. Analogously, it can be proved that $d_{T^k(D)}(y, x) \geq k$.

Claim 6. $H$ is a $(k - 1)$-absorbent set of vertices of $T^k(D)$. Clearly, $R^k(D)$ is an spanning subdigraph of $T^k(D)$ and we have proved (Theorem 2.3) that $H$ is a $k$-kernel of $R^k(D)$, in particular $H$ is a $(k - 1)$-absorbent set of vertices of $R^k(D)$. Thus $H$ is a $(k - 1)$-absorbent set of vertices of $T^k(D)$.

Observe that the set of black vertices in Figs. 1 and 2 is a 3-kernel.

Remark 2.1. It is easy to prove that for $D = \bar{C}_4$ (the directed cycle of length 4) and $k = 2$, the $k$-total digraph of $D$, $T^k(D)$ has no $k$-kernel. Thus the assertion given in Theorem 2.4 cannot be improved.

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References


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