ON COVARIETY LATTICES

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Abstract

This paper shows basic properties of covariety lattices. Such lattices are shown to be infinitely distributive. The covariety lattice \( L_{CV}(K) \) of subcovarieties of a covariety \( K \) of \( F \)-coalgebras, where \( F : \text{Set} \rightarrow \text{Set} \) preserves arbitrary intersections is isomorphic to the lattice of subcoalgebras of a \( \mathcal{P}_\kappa \)-coalgebra for some cardinal \( \kappa \). A full description of the covariety lattice of \( Id \)-coalgebras is given. For any topology \( \tau \) there exist a bounded functor \( F : \text{Set} \rightarrow \text{Set} \) and a covariety \( K \) of \( F \)-coalgebras, such that \( L_{CV}(K) \) is isomorphic to the lattice \( (\tau, \cup, \cap) \) of open sets of \( \tau \).

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1. Introduction

Many mathematicians and computer scientists have been recently studying the universal theory of coalgebras - objects dual to algebras. Many interesting properties of coalgebras have been shown. E.g. an analogue of the Birkhoff Variety Theorem was developed, which describes syntactically the classes of coalgebras called covarieties.

This paper studies the basic properties of covariety lattices. We show that the covariety lattices are infinitely distributive. Corollary 3.9 shows
that given any $F$-coalgebra $A$ there is a covariety $K$ of $A \times F$-coalgebras such that the lattice $L_{CV}(K)$ of subcovarieties of $K$ is isomorphic to the lattice $S(A)$ of subcoalgebras of $A$.

Next, Theorem 4.1 shows that, whenever $F$ preserves arbitrary intersections, the covariety lattice is isomorphic to the lattice $D(R_{Set^F})$ of subsets of all rooted coalgebras closed under taking rooted subcoalgebras of homomorphic images. As an example, the covariety lattice of $Id$-coalgebras is described.

Finally, the covariety lattice $L_{CV}(K)$ of subcovarieties of a covariety $K$ of $F$-coalgebras, where $F : Set \rightarrow Set$ preserves arbitrary intersections, is characterized in Theorem 4.5 as the lattice of subcoalgebras of some $P^\kappa$-coalgebra.

2. Basic definitions and properties

Let $Set$ be the category of all sets and mappings between them. Let $F : Set \rightarrow Set$ be a functor. An $F$-coalgebra $A$ is a pair $(A, \alpha)$, where $A$ is a set and $\alpha$ is a mapping $\alpha : A \rightarrow F(A)$. The set $A$ is called the carrier of the coalgebra $(A, \alpha)$ and the mapping $\alpha$ is called the structure.

Let $A = (A, \alpha)$ and $B = (B, \beta)$ be two $F$-coalgebras. A homomorphism from the coalgebra $A$ to the coalgebra $B$ is a mapping $h : A \rightarrow B$, such that $F(h) \circ \alpha = \beta \circ h$.

The class of all $F$-coalgebras together with homomorphisms as morphisms forms a category denoted by $Set^F$. An $F$-coalgebra $B$ is said to be a homomorphic image of an $F$-coalgebra $A$ if there exists a surjective homomorphism from $A$ onto $B$. An $F$-coalgebra $S$ is said to be a subcoalgebra of an $F$-coalgebra $A$ if there exists an injective homomorphism from $S$ into $A$. This is denoted by $S \leq A$.

**Theorem 2.1** [2]. Let $F : Set \rightarrow Set$ be a functor. Let $\{S_i\}_{i \in I}$ be a family of subcoalgebras of an $F$-coalgebra $A$. Then

- there exists a unique structure $\alpha : \bigcup_{i \in I} S_i \rightarrow F(\bigcup_{i \in I} S_i)$ such that the coalgebra $\bigcup_{i \in I} S_i := (\bigcup_{i \in I} S_i, \alpha)$ is a subcoalgebra of $A$;

- if $I$ is a finite set of indices, then there exists a unique structure $\beta : \bigcap_{i \in I} S_i \rightarrow F(\bigcap_{i \in I} S_i)$ such that $\bigcap_{i \in I} S_i := (\bigcap_{i \in I} S_i, \beta)$ is a subcoalgebra of $A$.

In other words, subcoalgebras of a given coalgebra form a topology.
Theorem 2.2 [2]. Let $F : \text{Set} \to \text{Set}$ be a functor and $\mathbb{A}$ be an $F$-coalgebra. If $S \subseteq A$, then there exists at most one structure $\sigma : S \to F(S)$ such that $(S, \sigma) \leq \mathbb{A}$.

The disjoint union of a family $\{X_j\}_{j \in J}$ of sets is denoted by $\Sigma_{j \in J} X_j$. Now let $\{\mathbb{A}_i\}_{i \in I}$ be a family of $F$-coalgebras. The disjoint sum $\Sigma_{i \in I} \mathbb{A}_i$ of the family $\{\mathbb{A}_i\}_{i \in I}$ of $F$-coalgebras is an $F$-coalgebra defined as follows: The carrier set of the disjoint sum $\mathbb{A} = \Sigma_{i \in I} \mathbb{A}_i$ is the disjoint union of the carriers of $\mathbb{A}_i$, i.e.

$$A := \Sigma_{i \in I} A_i.$$ 

The structure $\alpha : A \to F(A)$ of the disjoint sum $\mathbb{A} = \Sigma_{i \in I} \mathbb{A}_i$ is defined as follows:

$$\alpha : A \to F(A); A_i \ni a \mapsto F(e_i) \circ \alpha_i(a),$$

where the mapping $\alpha_i$ denotes the structure of the coalgebra $\mathbb{A}_i$ and $e_i : A_i \to A; a \mapsto (a, i)$, for every $i \in I$. We say that an $F$-coalgebra $\mathbb{A}$ is a conjunct sum of the family $\{G_i\}_{i \in I}$ of $F$-coalgebras if there exists a family $\{e_i : G_i \to \mathbb{A}\}_{i \in I}$ of injective homomorphisms such that $A = \bigcup_{i \in I} e_i(G_i)$. We denote it by $\mathbb{A} \in \Sigma^I(\{G_i\}_{i \in I})$.

A functor $F : \text{Set} \to \text{Set}$ is said to preserve arbitrary intersections if for any family of subcoalgebras $\{\mathbb{A}_i\}_{i \in I}$ of an $F$-coalgebra $\mathbb{A}$, there exists a structure $\alpha : \bigcap A_i \to F(\bigcap A_i)$ such that the $F$-coalgebra $\bigcap A_i := (\bigcap A_i, \alpha)$ is a subcoalgebra of $\mathbb{A}$.

A functor $F : \text{Set} \to \text{Set}$ is said to be bounded by $\kappa$, if $\kappa$ is the cardinal number such that for every $F$-coalgebra $\mathbb{A}$ and for every $a \in A$ there exists an $F$-coalgebra $U_a$, such that $|U_a| \leq \kappa$, $a \in U_a$ and $U_a \subseteq \mathbb{A}$. We say that $F$ is bounded if it is bounded by $\kappa$ for some cardinal $\kappa$.

Example 2.3. Let $\kappa$ be a cardinal number. Let $\mathcal{P}_\kappa : \text{Set} \to \text{Set}$ be the functor given by $\mathcal{P}_\kappa(X) = \{S \subseteq X \mid |S| \leq \kappa\}$ for a set $X$ and

$$\mathcal{P}_\kappa(f) : \mathcal{P}_\kappa(X) \to \mathcal{P}_\kappa(Y); S \mapsto f(S)$$

for a mapping $f : X \to Y$. The functor $\mathcal{P}_\kappa$ is an example of a bounded functor which preserves arbitrary intersections (see [5]).
Example 2.4. The filter functor $\mathcal{F} : \text{Set} \to \text{Set}$ assigns to every set $X$ the set of filters $\mathcal{F}(X)$ on $X$ and to every mapping $f : X \to Y$ the mapping

$$
\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y); \quad F \mapsto \{f(W) \mid W \in F\},
$$

where $\uparrow \{f(W) \mid W \in F\}$ denotes the filter generated by the set $\{f(W) \mid W \in F\}$. This functor is an example of a functor which does not preserve arbitrary intersections.

It is important to mention that any topological space can be turned into an $\mathcal{F}$-coalgebra. Let $(X, \tau)$ be a topological space. Define the mapping

$$
\sigma : X \to \mathcal{F}(X); \quad x \mapsto \{W \subseteq X \mid \exists O \in \tau \text{ such that } x \in O \subseteq W\}.
$$

The subcoalgebras of the $\mathcal{F}$-coalgebra $(X, \sigma)$ are precisely the open subsets of $(X, \tau)$ (see [3]). Since the intersection of an arbitrary family of open sets in a given topological space may not exist, it is clear that $\mathcal{F}$ does not preserve arbitrary intersections. The filter functor $\mathcal{F}$ is not bounded.

Let $K$ be a class of $\mathcal{F}$-coalgebras. We define the following classes of $\mathcal{F}$-coalgebras:

$$
\mathcal{S}(K) := \{S \mid \exists A \in K \text{ such that } S \subseteq A\},
$$

$$
\mathcal{H}(K) := \{B \mid \exists A \in K \text{ such that } A \to B\},
$$

$$
\Sigma(K) := \{\Sigma_{i \in I} A_i \mid \{A_i\}_{i \in I} \subseteq K\}.
$$

A class $K$ of $\mathcal{F}$-coalgebras is called a covariety if it is closed under $\mathcal{S}, \mathcal{H}$ and $\Sigma$, i.e., $\mathcal{S}(K) \subseteq K$, $\mathcal{H}(K) \subseteq K$ and $\Sigma(K) \subseteq K$.

Theorem 2.5 [2]. Let $K$ be a class of $\mathcal{F}$-coalgebras. The class $\mathcal{S}\mathcal{H}\Sigma(K)$ is the smallest covariety containing $K$.

We say that a class $K'$ of $\mathcal{F}$-coalgebras is a subcovariety of a covariety $K$ whenever $K'$ is a covariety and $K' \subseteq K$.

The assumption of boundedness of a functor $F$ guarantees that the collection of all subcovarieties of the covariety $\text{Set}_F$ is a set (see [2]). Since we do not want to focus only on coalgebras for bounded functors we need to allow class based lattices, i.e., partially ordered classes in which each pair of elements has a supremum and an infimum. Obviously, any lattice is a class based lattice. We may easily generalize the notion of completeness to the
class based lattices. Namely, a partially ordered class \((C, \leq)\) is a complete class based lattice if all its subclasses have a supremum and infimum. We see that whenever \((C, \leq)\) is a complete class based lattice and \(C\) is a proper set then \((C, \leq)\) is simply a complete lattice. The following holds.

**Theorem 2.6.** *The collection of all subcovarieties of a given covariety \(K\) of \(F\)-coalgebras ordered by inclusion is a complete class based lattice.*

We denote the class based lattice of all subcovarieties of \(K\) by \(L_{CV}(K)\). Let \(\{K_i\}_{i \in I}\) be a collection of subcovarieties of the covariety \(K\) of \(F\)-coalgebras. Note that the collection \(\{K_i\}_{i \in I}\) and hence \(I\) may be a proper class. The infimum and supremum of \(\{K_i\}_{i \in I}\) in \(L_{CV}(K)\) are of the following form.

\[
\prod_{i \in I} K_i := \bigcap_{i \in I} K_i,
\]

\[
\sum_{i \in I} K_i := \mathcal{H}\Sigma \left( \bigcup_{i \in I} K_i \right).
\]

We will clearly distinguish between the class based lattices whose carrier is a proper class and lattices with a set carrier. We will use the term *proper lattice* to emphasize the fact that the latter holds, i.e. a class based lattice is simply a lattice.

### 3. Covariety lattices

In this section we discuss the distributivity of covariety class based lattices. Then we describe the lattices \(L_{CV}(SH\Sigma(A))\) for certain coalgebras \(A\) and show that the lattice of open sets of any topological space is isomorphic to some covariety lattice \(L_{CV}(K)\).

Suppose \(F\) is bounded by \(|X|\) for some set \(X\). Then the cofree \(F\)-coalgebra \(C_X\) over the set \(X\) exists. In this case there is a one-to-one correspondence between the so-called invariant subcoalgebras of \(C_X\) and covarieties of \(F\)-coalgebras. This correspondence is given by the following formula:

\[
K = Q(C_X, U) := \{A \mid \forall \phi : A \to C_X, \phi(A) \subseteq U\},
\]

where \(U := \bigcup \{\phi(A) \mid \phi : A \to C_X\text{ and }A \in K\}\) (see [2]). Therefore, the lattice \(L_{CV}(\text{Set}_F)\) of all covarieties of \(F\)-coalgebras is isomorphic to the lattice

\[\quad \]
of invariant subcoalgebras of $C_X$ ordered by inclusion. Because it is clear that the invariant subcoalgebras are closed under infinite unions and finite intersections the lattice $L_{CV}(\text{Set}_F)$ is infinitely distributive. If we do not assume boundedness of $F$ then we cannot speak of the above correspondence. Yet, we are able to derive the following result directly.

**Theorem 3.1.** The class based lattice $L_{CV}(\text{Set}_F)$ of covarieties of $F$-coalgebras is distributive.

**Proof.** Let $\{K_i\}_{i \in I}$ be a collection of covarieties of $F$-coalgebras and let $K$ be a covariety. Note that $I$ may be a proper class. To show that the covariety class based lattice $L_{CV}(\text{Set}_F)$ is distributive it is enough to verify that the following inequality is true:

$$K \cdot \left( \sum_{i \in I} K_i \right) \leq \sum_{i \in I} K \cdot K_i.$$  

Let $\mathbb{A} \in K \cdot \left( \sum_{i \in I} K_i \right)$. This means that $\mathbb{A} \in K$ and $\mathbb{A} \in \sum_{i \in I} K_i$. Since $\sum_{i \in I} K_i = \text{SH}(\bigcup_{i \in I} K_i)$, it follows that $\mathbb{A} \leq h(\sum_{j \in J} B_j)$, where $B_j \in \bigcup_{i \in I} K_i$ for any $j$ coming from the set of indices $J$ and $h$ is a homomorphism. Let $e_k : B_k \rightarrow \sum_{j \in J} B_j$ for $k \in J$ denote the canonical embeddings. Then $\mathbb{A} \leq \bigcup_{j \in J} h(e_j(B_j))$. By Theorem 2.1 we have

$$\mathbb{A} = \bigcup_{j \in J} h(e_j(B_j)) \cap \mathbb{A}.$$

Since all $K_i$’s are covarieties and $h(e_j(B_j)) \cap \mathbb{A} \leq h(e_j(B_j))$, it follows that $h(e_j(B_j)) \cap \mathbb{A} \in K_j$ for some $i_j \in I$. Moreover, because $h(e_j(B_j)) \cap \mathbb{A} \leq \mathbb{A}$ and $\mathbb{A} \in K$, we have $h(e_j(B_j)) \cap \mathbb{A} \in K$. Hence $h(e_j(B_j)) \cap \mathbb{A} \in K \cdot K_j$ and therefore

$$\mathbb{A} \in \sum_{i \in I} K \cdot K_i.$$

**Definition 3.2** ([2]). An $F$-coalgebra $\mathbb{A}$ is called strongly simple whenever it does not possess any nontrivial homomorphic images.

We will now show some properties of strongly simple coalgebras, necessary for characterisation of $L_{CV}(\text{SH}(\mathbb{A}))$. 


Lemma 3.3 ([2]). Let $\mathbb{A}$ be a strongly simple $F$-coalgebra. If $B$ is an $F$-coalgebra, then there exists at most one homomorphism $h : B \to \mathbb{A}$.

Lemma 3.4. Let $\mathbb{A} = (A, \alpha)$ be a strongly simple $F$-coalgebra. Let $S \leq T \leq \mathbb{A}$ be such that $S \cong T$. Then $S = T$.

Lemma 3.5. Let $\mathbb{A}$ be a strongly simple $F$-coalgebra. If $B \in \mathcal{SH}(\mathbb{A})$, then $B \in \Sigma^C S(\mathbb{A})$.

**Proof.** If $B \in \mathcal{SH}(\mathbb{A})$ then $B \leq h(\Sigma_{i \in I} \mathbb{A})$, where $h$ is a homomorphism. Let $e_i : \mathbb{A} \to \Sigma_{i \in I} \mathbb{A}$ denote the canonical embeddings. Since $\mathbb{A}$ is strongly simple, it follows that the image coalgebra $h(e_i(\mathbb{A}))$ is isomorphic to $\mathbb{A}$ for each $i \in I$. Since $h(\Sigma_{i \in I} \mathbb{A}) = \bigcup_{i \in I} h(e_i(\mathbb{A}))$, it follows that $B = \bigcup_{i \in I} h(e_i(\mathbb{A})) \cap B$. Because $h(e_i(\mathbb{A})) \cap B \leq h(e_i(\mathbb{A})) \cong \mathbb{A}$, we have $B \in \Sigma^C S(\mathbb{A})$. $\square$

Let $\mathbb{A}$ be an $F$-coalgebra. Let $S(\mathbb{A})$ denote the set of carriers of subcoalgebras of $\mathbb{A}$, i.e.

$$S(\mathbb{A}) := \{ B \mid B \leq \mathbb{A} \}.$$  

By Theorem 2.1, the set $S(\mathbb{A})$ together with the operations of union and intersection forms a lattice.

What we now want to do is to show without any additional assumptions that $L_{CV}(\mathcal{SH}(\mathbb{A}))$ is isomorphic to the proper lattice $(S(\mathbb{A}), \cup, \cap)$ for any strongly simple coalgebra $\mathbb{A}$. If we assume that $F$ is bounded then the cofree $F$-coalgebra $C_1$ over the one-element set 1 exists. The coalgebra $C_1$ is the terminal object in the category $\text{Set}_F$. Therefore, it is strongly simple. Moreover, strongly simple $F$-coalgebras are precisely subcoalgebras of $C_1$ and all subcoalgebras of $C_1$ are invariant. Hence, $L_{CV}(\mathcal{SH}(C_1))$ is isomorphic to $(S(C_1), \cup, \cap)$. The same thing is clearly true for any subcoalgebra of $C_1$. If we do not assume that $F$ is bounded the terminal object in $\text{Set}_F$ may not exist. Yet, we can expand our category $\text{Set}_F$ to class based coalgebras, where the terminal object always exists (see [1]). Using a similar argument and working with class based coalgebras and we get a general result. At the same time if one does not prefer to work with classes then the direct proof of the following theorem is an alternative.

Theorem 3.6. Let $\mathbb{A}$ be a strongly simple $F$-coalgebra. Then $L_{CV}(\mathcal{SH}(\mathbb{A}))$ is a proper lattice and

$$L_{CV}(\mathcal{SH}(\mathbb{A})) \cong (S(\mathbb{A}), \cup, \cap).$$
**Proof.** Let $K$ be a subcovariety of the covariety $SH\Sigma(A)$. Define

$$S_K := \bigcup \{S | S \leq A \text{ and } S \in K\}.$$ 

In other words, the $F$-coalgebra $S_K$ is the union of subcoalgebras of $A$ which are elements of the covariety $K$.

It is clear that $S_K$ is the greatest subcoalgebra of $A$ contained in $K$.

Let $B \in K$. We have $B \leq {f(\Sigma_i \in I A)}$ for a homomorphism $f$. By Lemma 3.5, $B \in \Sigma^C(\{C_i\}_{i \in I})$, where $C_i \leq A$ for $i \in I$. Since $C_i \leq B$, it follows that $C_i \in K$. Hence $C_i \leq S_K$ for $i \in I$ and $B \in \Sigma^C S(S_K)$. Therefore any coalgebra $B \in K$ is a conjunct sum of subcoalgebras of $S_K$, i.e. $K = \Sigma^C S(S_K)$.

We will now prove that the mapping

$$S(-) : \text{Lcv}(SH\Sigma(A)) \rightarrow \text{S}(A); K \mapsto S_K$$

is a lattice isomorphism. To show that it is injective, let $K_1$ and $K_2$ be subcovarieties of the covariety $SH\Sigma(A)$ such that $S_{K_1} = S_{K_2}$. Then

$$K_1 = \Sigma^C S(S_{K_1}) = \Sigma^C S(S_{K_2}) = K_2.$$ 

We will now show that $S(-)$ is a surjection. Let $C \leq A$. Then

$$C \leq S_{SH\Sigma(C)}.$$ 

Since $S_{SH\Sigma(C)} \leq A$ and since $S_{SH\Sigma(C)} \in SH\Sigma(C) = \Sigma^C S(C)$, it follows that

$$S_{SH\Sigma(C)} \in \Sigma^C(\{D_j\}_{j \in J}),$$

where $D_j \leq C$. This means that for any $j \in J$, the coalgebra $S_{SH\Sigma(C)}$ contains a coalgebra $D_j$ isomorphic to $D_j$ as its subcoalgebra. Hence

$$\bar{D}_j \leq S_{SH\Sigma(C)} \leq A$$

for all $j \in J$, and $\bar{D}_j \leq C \leq A$. By Lemma 3.4, we have $\bar{D}_j = D_j$. Therefore,

$$S_{SH\Sigma(C)} = \bigcup_{j \in J} \bar{D}_j = \bigcup_{j \in J} D_j \leq C$$

and $S_{SH\Sigma(C)} = C$. Consequently the mapping $S(-)$ is a bijection. Since it is clear that $S(-)$ is order preserving we immediately get that $S(-)$ is the isomorphism from the lattice $\text{Lcv}(SH\Sigma(A))$ onto $(\text{S}(A), \cup, \cap)$.  

$\blacksquare$
For an $F$-coalgebra $A = (A, \alpha)$ and a set $B$ such that $A \subseteq B$, we define the following $B \times F$-coalgebra:

$$A_B := (A, (\subseteq B, \alpha)).$$

The structure map of $A_B$ is the following:

$$(\subseteq B, \alpha) : A \rightarrow B \times F(A); a \mapsto (a, \alpha(a)).$$

This easy trick allows us to force the $B \times F$-coalgebra $A_B$ to be strongly simple and at the same time to leave the subcoalgebras of $A$ untouched. This property is formally described by the following lemmata.

**Lemma 3.7.** Let $A = (A, \alpha)$ be an $F$-coalgebra and let $B$ be a set such that $A \subseteq B$. Then the $B \times F$-coalgebra $A_B$ is strongly simple.

**Lemma 3.8.** Let $A = (A, \alpha)$ be an $F$-coalgebra and let $B$ be a set such that $A \subseteq B$. Then $(S(A), \cup, \cap) = (S(A_B), \cup, \cap)$.

**Corollary 3.9.** Let $(X, \tau)$ be a topological space. There exists a bounded functor $F : \text{Set} \rightarrow \text{Set}$ and a covariety $K$ of $F$-coalgebras such that $L_{CV}(K)$ is isomorphic to the lattice $(\tau, \cup, \cap)$ of open sets in $\tau$.

**Proof.** It follows by Example 2.4, Lemma 3.8, Lemma 3.7 and Theorem 3.6.

**4. Covariety lattices for functors preserving arbitrary intersections**

Throughout this section we will assume that $F$ is a bounded functor. Therefore, the collection of all covarieties of $F$-coalgebras is a set. It is worth noting that almost all of the results presented here naturally generalize to the case when classes of covarieties are allowed.

Given a strongly simple $F$-coalgebra $A$, Theorem 3.6 describes the lattice of subcovarieties of the covariety $\mathcal{SH}(\Sigma(A))$ in terms of the lattice of subcoalgebras of $A$. The following question arises: can we describe the covariety lattice of any covariety $K$ of $F$-coalgebras in a similar way in terms of subcoalgebras of an $F$-coalgebra? In general the answer is “no”, which is seen in the Example 4.4. But first, we will characterize the lattice $L_{CV}(\text{Set}_F)$ in the case the functor $F$ preserves arbitrary intersections.
An \( F \)-coalgebra \( \mathbb{A} \) is called \emph{rooted} (or \emph{one-generated}) if there exists an element \( a \in A \), called a \emph{root}, such that the coalgebra \( \mathbb{A} \) is the smallest subcoalgebra of \( A \) containing the element \( a \). If \( a \in A \) is a root of a rooted coalgebra \( \mathbb{A} \), then we say that \( \mathbb{A} \) is generated by \( a \).

If \( F : \text{Set} \to \text{Set} \) preserves arbitrary intersections, then all rooted \( F \)-coalgebras are of the following form

\[
\langle a \rangle := \bigcap \{ S \mid a \in S \text{ and } S \subseteq \mathbb{A} \},
\]

for some \( F \)-coalgebra \( \mathbb{A} \) and \( a \in A \). For any \( F \)-coalgebra \( \mathbb{A} \), we have \( \mathbb{A} = \bigcup_{a \in A} \langle a \rangle \). It follows that \( \mathbb{A} \in \Sigma^C(\{ \langle a \rangle \}_{a \in A}) \).

Let \( K \) be a class of \( F \)-coalgebras. Let \( \mathcal{R}_K \) denote the collection of rooted \( F \)-coalgebras consisting of exactly one representative from each class of isomorphic rooted \( F \)-coalgebras from the class \( K \). If \( \mathbb{A}, \mathbb{B} \in \mathcal{R}_K \) and are isomorphic, then \( \mathbb{A} = \mathbb{B} \). By the assumption of boundedness of \( F \) we know that \( \mathcal{R}_K \) is a proper set. Let \( D(\mathcal{R}_K) \) denote the set of subsets of \( \mathcal{R}_K \) closed under taking subcoalgebras of homomorphic images, i.e.:

\[
D(\mathcal{R}_K) := \{ U \subseteq \mathcal{R}_K \mid \mathcal{R}_K \cap SH(U) = U \}.
\]

**Theorem 4.1.** If \( F : \text{Set} \to \text{Set} \) preserves arbitrary intersections then the lattice \( L_{\text{Cov}}(\text{Set}_F) \) of subcovarieties of \( \text{Set}_F \) is isomorphic to the lattice \( (D(\mathcal{R}_{\text{Set}_F}), \cup, \cap) \).

**Proof.** Let \( K \) be a covariety of \( F \)-coalgebras. Let \( \mathbb{A} \in \mathcal{R}_K \). Then \( \mathbb{A} \) is a rooted coalgebra in the covariety \( K \). The rooted subcoalgebras of homomorphic images of \( \mathbb{A} \) are elements of the set \( \mathcal{R}_K \). This means that \( \mathcal{R}_K \in D(\mathcal{R}_{\text{Set}_F}) \). We define the following mapping.

\[
r : L_{\text{Cov}}(\text{Set}_F) \to D(\mathcal{R}_{\text{Set}_F}); K \mapsto \mathcal{R}_K.
\]

We will show that \( r \) is an isomorphism. Let \( K_1 \) and \( K_2 \) be two covarieties such that \( r(K_1) = r(K_2) \). Let \( \mathbb{A} \in K_1 \). For any \( a \in A \) the rooted coalgebra \( \langle a \rangle \) is a subcoalgebra of \( \mathbb{A} \). Hence \( \langle a \rangle \in K_1 \) and \( \langle a \rangle \in K_2 \). Since \( \mathbb{A} = \bigcup_{a \in A} \langle a \rangle \), the coalgebra \( \mathbb{A} \) belongs to \( K_2 \). Therefore, \( K_1 = K_2 \) and the mapping \( r \) is injective.

Now let \( U \in D(\mathcal{R}_{\text{Set}_F}) \). The smallest covariety containing \( U \) is given by the class \( SH(U) \). It is clear that \( U \subseteq r(SH(U)) \). Now let \( \mathbb{A} \in r(SH(U)) \). This means that \( \mathbb{A} \) is a rooted coalgebra, say \( \mathbb{A} = \langle a \rangle \), and
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is a subcoalgebra of \(B\), where \(B = h(\Sigma_{i \in I} C_i)\) is a homomorphic image of the disjoint sum of a family \(\{C_i\}_{i \in I}\) of rooted coalgebras in \(U\). Let \(e_i : C_i \rightarrow \Sigma_{i \in I} C_i\) denote the canonical embeddings. It is easy to see that \(B = h(\Sigma_{i \in I} C_i) = \bigcup_{j \in I} h(e_i(C_i))\). Since \(a \leq B\), it follows that \(a \in h(e_j(C_j))\) for some \(j \in I\). Hence \(a \leq h(e_j(C_j))\). Since \(U\) is closed under taking rooted subcoalgebras of homomorphic images, it follows that \(A = \langle a \rangle \in U\). Therefore \(U = r(\mathcal{SH}(U))\) and the mapping \(r\) is surjective. Consequently \(r\) is bijective. It is clear that the mapping \(r\) is an order embedding. Hence \(r\) is a lattice isomorphism.

Remark 4.2. It is worth noting that the mapping \(r\) in the proof of Theorem 4.1 is in fact a complete lattice isomorphism.

Corollary 4.3. Let \(F : \text{Set} \rightarrow \text{Set}\) preserve arbitrary intersections and let \(K\) be a covariety of \(F\)-coalgebras. Then \(L_{CV}(K) \cong (\mathcal{D}(\mathcal{R}_K), \cup, \cap)\).

Example 4.4. We will describe the covariety lattice \(L_{CV}(\text{Set}_{\text{Id}})\). By Theorem 4.1, the first step is to find all rooted \(\text{Id}\)-coalgebras. Note that \(\text{Id}\)-coalgebras are exactly mono-unary algebras. Therefore, we can speak of an index and period of a given coalgebra. E.g. \((0, 2)\) denotes the coalgebra given by the diagram \(\bullet \Rightarrow \bullet\) and \((1, 2)\) by the diagram \(\bullet \rightarrow \bullet \Rightarrow \bullet\). Given a finite rooted \(\text{Id}\)-coalgebra \((i, p)\), it is not hard to notice that any rooted subcoalgebra of \((i, p)\) is of the form \((i', p)\), where \(i' \leq i\). Any subcoalgebra of \((\infty, 0)\) is of the form \((\infty, 0)\). Moreover, any rooted homomorphic image of a coalgebra \((i, p)\) is of the form \((i', p')\), where \(i' \leq i\) and \(p' | p\). Therefore,

\[
\mathcal{SH}((i, p)) = \{(i', p') \in N_0 \times N \cup \{(\infty, 0)\} \mid i' \leq i \text{ and } p' | p\}.
\]

We can introduce a partial order on \(N_0 \times N \cup \{(\infty, 0)\}\) as follows: \((i', p') \preceq (i, p) : \iff i' \leq i\) and \(p' | p\). Then

\[
\mathcal{SH}((i, p)) = \downarrow (i, p) := \{(i', p') \mid (i', p') \preceq (i, p)\}.
\]

By Theorem 4.1, the lattice \(L_{CV}(\text{Set}_{\text{Id}})\) of subcovarieties of \(\text{Set}_{\text{Id}}\) is isomorphic to the lattice of downsets \((\mathcal{O}(N_0 \times N \cup \{(\infty, 0)\}), \cup, \cap)\) of the poset \(N_0 \times N \cup \{(\infty, 0)\}\).
Now, consider the \( \text{Id} \)-coalgebra \((1, 2)\). The covariety lattice of \( \mathcal{SH}(1, 2) \) looks as follows:

\[
\begin{array}{c}
\mathcal{SH}(1, 2) \\
\downarrow \\
\mathcal{SH}(0, 2) \\
\downarrow \\
\mathcal{SH}(0, 1) \\
\downarrow \\
\emptyset
\end{array}
\]

At the beginning of this section we stated a question whether it was possible to describe a covariety lattice \( L_{CV}(K) \) of any covariety \( K \) of \( F \)-coalgebras in terms of subcoalgebras of an \( F \)-coalgebra. We will show that it is impossible to construct an \( \text{Id} \)-coalgebra \( A \), whose subcoalgebra lattice is isomorphic to the covariety lattice \( \mathcal{SH}(1, 2) \). By contradiction, assume that there exists \( \text{Id} \)-coalgebra \( A \) whose subcoalgebra lattice is the following:

\[
\begin{array}{c}
A \\
B \cup C \\
B \\
B \cap C \\
\emptyset
\end{array}
\]

Join irreducible elements, i.e. \( B, C \) and \( B \cap C \), must be rooted \( \text{Id} \)-coalgebras. The rooted coalgebra \( B \cap C \) does not contain any proper subcoalgebras. This means that \( B \cap C \) is a cycle. The coalgebras \( B = \langle b \rangle \) and \( C = \langle c \rangle \) cover the coalgebra \( B \cap C \). Hence the coalgebra \( B \cup C \) has the following form.

\[
\begin{array}{c}
b \\
\downarrow \\
\bullet - \cdots - \bullet
\end{array}
\quad
\begin{array}{c}
c \\
\downarrow \\
\bullet - \cdots - \bullet
\end{array}
\]

Since \( A \) itself is join irreducible, it follows that it is rooted, i.e. \( A = \langle a \rangle \). On one hand the element \( a \) has to be connected directly with the element \( b \) and on the other with the element \( c \), which is a contradiction. \( \square \)
Theorem 4.5. Let \( F : \text{Set} \to \text{Set} \) be a functor preserving arbitrary intersections. Then the lattice \( L_{CV}(K) \) of subcovarieties of a covariety \( K \) of \( F \)-coalgebras is isomorphic to the lattice of subcoalgebras of some \( P_\kappa \)-coalgebra.

Conversely, for any \( P_\kappa \)-coalgebra \( \mathbb{A} \), there exists a functor \( F : \text{Set} \to \text{Set} \) preserving arbitrary intersections and a covariety \( K \) of \( F \)-coalgebras such that the lattice \( L_{CV}(K) \) is isomorphic to the lattice of subcoalgebras of \( \mathbb{A} \).

**Proof.** If \( F \) preserves arbitrary intersection, then by Theorem 4.1, the lattice \( L_{CV}(\text{Set}_F) \) of subcovarieties of \( \text{Set}_F \) is isomorphic to the lattice \( (D(\mathbb{R}_{\text{Set}_F}), \cup, \cap) \). Take \( \kappa := |\mathbb{R}_{\text{Set}_F}| \). Define a \( P_\kappa \)-coalgebra \( (\mathbb{R}_{\text{Set}_F}, \eta) \) as follows. For \( \langle a \rangle \in \mathbb{R}_{\text{Set}_F} \), define
\[
\eta(\langle a \rangle) := \mathcal{SH}(\langle a \rangle) \cap \mathbb{R}_{\text{Set}_F}.
\]
Then clearly
\[
\mathcal{S}(\langle \mathbb{R}_{\text{Set}_F}, \eta \rangle) \cong D(\mathbb{R}_{\text{Set}_F}) \cong L_{CV}(\text{Set}_F).
\]
Conversely let \( \mathbb{A} = (A, \alpha) \) be a \( P_\kappa \)-coalgebra. Then by Theorem 3.6, the lattice \( L_{CV}(\mathcal{SH}(A)) \) of subcovarieties of the covariety \( \mathcal{SH}(A) \) of \( A \times P_\kappa \)-coalgebras is isomorphic to \( \mathcal{S}(\mathbb{A}) \) and the functor \( A \times P_\kappa \) is bounded and preserves arbitrary intersections.

**References**


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