WREATH PRODUCT OF A SEMIGROUP AND A $\Gamma$-SEMIGROUP

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Abstract

Let $S = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two nonempty sets. $S$ is called a $\Gamma$-semigroup if $aob \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and $(aob)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. In this paper we study the semidirect product of a semigroup and a $\Gamma$-semigroup. We also introduce the notion of wreath product of a semigroup and a $\Gamma$-semigroup and investigate some interesting properties of this product.

Keywords: semigroup, $\Gamma$-semigroup, orthodox semigroup, right(left) orthodox $\Gamma$-semigroup, right(left) inverse semigroup, right(left) inverse $\Gamma$-semigroup, right(left)$\alpha$- unity, $\Gamma$-group, semidirect product, wreath product.

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1. Introduction

The notion of a $\Gamma$-semigroup has been introduced by Sen and Saha [7] in the year 1986. Many classical notions of semigroup have been extended to $\Gamma$-semigroup. In [1] and [2] we have introduced the notions of right inverse $\Gamma$-semigroup and right orthodox $\Gamma$-semigroup. In [6] we have studied the semidirect product of a monoid and a $\Gamma$-semigroup as a generalization of [4] and [5]. We have obtained necessary and sufficient conditions for a semidirect product of the monoid and a $\Gamma$-semigroup to be right (left) orthodox $\Gamma$-semigroup and right (left) inverse $\Gamma$-semigroup. In [9] Zhang has studied the semidirect product of semigroups and also studied wreath product of semigroups. In this paper we generalize the results of Zhang to the semidirect product of a semigroup and a $\Gamma$-semigroup. We also study the wreath product of a semigroup and a $\Gamma$-semigroup.

2. Preliminaries

We now recall some definitions and results relating our discussion.

**Definition 2.1.** Let $S = \{a, b, c, \ldots \}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots \}$ be two nonempty sets. $S$ is called a $\Gamma$-semigroup if

(i) $a \alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and

(ii) $(a \alpha b) \beta c = a \alpha (b \beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Let $S$ be an arbitrary semigroup. Let 1 be a symbol not representing any element of $S$. We extend the binary operation defined on $S$ to $S \cup \{1\}$ by defining $11 = 1$ and $1a = a1 = a$ for all $a \in S$. It can be shown that $S \cup \{1\}$ is a semigroup with identity element 1. Let $\Gamma = \{1\}$. If we take $ab = a1b$, it can be shown that the semigroup $S$ is a $\Gamma$-semigroup where $\Gamma = \{1\}$. Thus a semigroup can be considered to be a $\Gamma$-semigroup.

Let $S$ be a $\Gamma$-semigroup and $x$ be a fixed element of $\Gamma$. We define $a \cdot b = axb$ for all $a, b \in S$. We can show that $(S, \cdot)$ is a semigroup and we denote this semigroup by $S_x$.

**Definition 2.2.** Let $S$ be a $\Gamma$-semigroup. An element $a \in S$ is said to be regular if $a \in a \Gamma S \Gamma a$ where $a \Gamma S \Gamma a = \{a \alpha b \beta a : b \in S, \alpha, \beta \in \Gamma\}$. $S$ is said to be regular if every element of $S$ is regular.
We now describe some examples of regular $\Gamma$-semigroup.

In [7] we find the following interesting example of a regular $\Gamma$-semigroup.

**Example 2.3.** Let $S$ be the set of all $2 \times 3$ matrices and $\Gamma$ be the set of all $3 \times 2$ matrices over a field. Then for all $A, B, C \in S$ and $P, Q \in \Gamma$ we have $APB \in S$ and since the matrix multiplication is associative, we have $(APB)QC = AP(BQC)$. Hence $S$ is a $\Gamma$-semigroup. Moreover it is regular shown in [7].

Here we give another example of a regular $\Gamma$-semigroup.

**Example 2.4.** Let $S$ be a set of all negative rational numbers. Obviously $S$ is not a semigroup under usual product of rational numbers. Let $\Gamma = \{ -\frac{1}{p} : p$ is prime $\}$. Let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Now if $aob$ is equal to the usual product of rational numbers $a, \alpha, b$, then $aob \in S$ and $(aob)\beta c = a\alpha(b\beta c)$. Hence $S$ is a $\Gamma$-semigroup. Let $a = \frac{m}{n} \in \Gamma$ where $m > 0$ and $n < 0$. $m = p_1p_2\ldots p_k$ where $p_i$’s are prime. $\beta = (\frac{1}{p_1})$ and $\alpha = (\frac{1}{p_k})$ we can say that $a$ is regular. Hence $S$ is a regular $\Gamma$-semigroup.

**Definition 2.5** [7]. Let $S$ be a $\Gamma$-semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an $\alpha$-idempotent if $eae = e$. The set of all $\alpha$-idempotents is denoted by $E_\alpha$. We denote $\bigcup_{\alpha \in \Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent elements of $S$.

**Definition 2.6** [7]. Let $a \in M$ and $\alpha, \beta \in \Gamma$. An element $b \in M$ is called an $(\alpha, \beta)$-inverse of $a$ if $a = aob\beta a$ and $b = b\beta aob$. In this case we write $b \in V^\beta_a(a)$.

**Definition 2.7** [2]. A regular $\Gamma$-semigroup $M$ is called a right (left) orthodox $\Gamma$-semigroup if for any $\alpha$-idempotent $e$ and $\beta$-idempotent $f$, $eaf$ (resp. $f\alpha e$) is a $\beta$-idempotent.

**Example 2.8** [2]. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. $S$ denotes the set of all mappings from $A$ to $B$. Here members of $S$ are described by the images of the elements 1, 2, 3. For example the map $1 \rightarrow 4, 2 \rightarrow 5, 3 \rightarrow 4$ is written as $(4, 5, 4)$ and $(5, 5, 4)$ denotes the map $1 \rightarrow 5, 2 \rightarrow 5, 3 \rightarrow 4$. A map from $B$ to $A$
is described in the same fashion. For example (1, 2) denotes $4 \rightarrow 1, 5 \rightarrow 2$. Now $S = \{(4, 4, 4), (4, 4, 5), (4, 5, 4), (4, 5, 5), (5, 4, 4), (5, 4, 5), (5, 5, 4), (5, 5, 5)\}$ and let $\Gamma = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$. Let $f, g \in S$ and $\alpha \in \Gamma$. We define $f \circ g$ by $(f \circ g)(a) = f(a)(g(a))$ for all $a \in A$. So $f \circ g$ is a mapping from $A$ to $B$ and hence $f \circ g \in S$ and we can show that $(f \circ g) \beta h = f(a)(g(\beta)h)$ for all $f, g, h \in S$ and $\alpha, \beta \in \Gamma$. We can show that each element $x$ of $S$ is an $\alpha$-idempotent for some $\alpha \in \Gamma$ and hence each element is regular. Thus $S$ is a regular $\Gamma$-semigroup. It is an idempotent $\Gamma$-semigroup. Moreover we can show that it is a right orthodox $\Gamma$-semigroup.

**Theorem 2.9** [2]. A regular $\Gamma$-semigroup $M$ is a right orthodox $\Gamma$-semigroup if and only if for $a, b \in M, V_\alpha^\beta(a) \cap V_\beta^\delta(b) \neq \emptyset$ for some $\alpha, \alpha_1, \beta \in \Gamma$ implies that $V_\alpha^\delta(a) = V_\alpha^\delta(b)$ for all $\delta \in \Gamma$.

**Definition 2.10** [1]. A regular $\Gamma$-semigroup is called a right (left) inverse $\Gamma$-semigroup if for any $\alpha$-idempotent $e$ and for any $\beta$-idempotent $f$, $e\alpha f \beta e = f \beta e$ ($e \beta f \alpha e = e \beta f$).

**Theorem 2.11** [7]. Let $S$ be a $\Gamma$-semigroup. If $S_\alpha$ is a group for some $\alpha \in S$ then $S_\alpha$ is a group for all $\alpha \in \Gamma$.

**Definition 2.12** [7]. A $\Gamma$-semigroup $S$ is called a $\Gamma$-group if $S_\alpha$ is a group for some $\alpha \in \Gamma$.

**Definition 2.13** [8]. A regular semigroup $S$ is said to be a right (left) inverse semigroup if for any $e, f \in E(S), ef = fe(efe = ef)$.

**Definition 2.14** [3]. A semigroup $S$ is called orthodox semigroup if it is regular and the set of all idempotents forms a subsemigroup.

**Definition 2.15** [7]. A nonempty subset $I$ of a $\Gamma$-semigroup $S$ is called a right (resp. left) ideal if $I \Gamma S \subseteq I$ (resp. $SI \subseteq I$). If $I$ is both a right ideal and a left ideal then we say that $I$ is an ideal of $S$.

**Definition 2.16** [7]. A $\Gamma$-semigroup $S$ is called right (resp. left) simple if it contains no proper right (resp. left) ideal i.e, for every $a \in S, a \Gamma S = S$ (resp. $S \Gamma a = S$). A $\Gamma$-semigroup is said to be simple if it has no proper ideals.

**Theorem 2.17** [7]. Let $S$ be a $\Gamma$- semigroup. $S$ is a $\Gamma$- group if and only if it is both left simple and right simple.
3. Semidirect Product of a Semigroup and a \( \Gamma \)-Semigroup

Let \( S \) be a semigroup and \( T \) be a \( \Gamma \)-semigroup. Let \( \text{End}(T) \) denote the set of all endomorphisms on \( T \) i.e., the set of all mappings \( f : T \to T \) satisfying \( f(ab) = f(a)f(b) \) for all \( a, b \in T, \alpha \in \Gamma \). Clearly \( \text{End}(T) \) is a semigroup. Let \( \phi : S \not\to \text{End}(T) \) be a given antimorphism i.e, \( \phi(rs) = \phi(r)\phi(s) \) for all \( r, s \in S \). If \( s \in S \) and \( t \in T \), we write \( t^s \) for \( (\phi(s))(t) \) and \( T^s = \{ t^s : t \in T \} \). Let \( S \times_\phi T = \{(s,t) : s \in S, t \in T \} \). We define \( (s_1, t_1) \alpha(s_2, t_2) = (s_1s_2, t_1^s t_2\alpha t_2) \) for all \( (s_1, t_1) \in S \times_\phi T \) and \( \alpha \in \Gamma \). Then \( S \times_\phi T \) is a \( \Gamma \)-semigroup. This \( \Gamma \)-semigroup \( S \times_\phi T \) is called the semidirect product of the semigroup \( S \) and the \( \Gamma \)-semigroup \( T \). In [6] we have studied the semidirect product \( S \times_\phi T \) assuming that \( S \) is a monoid. In this paper we investigate the properties of the semidirect product \( S \times_\phi T \) without taking \( 1 \) in \( S \).

**Lemma 3.1.** Let \( S \times_\phi T \) be a semidirect product of a semigroup \( S \) and a \( \Gamma \)-semigroup \( T \). Then

(i) \( (tau)^s = t^s au^s \) for all \( s \in S, t, u \in T \) and \( \alpha \in \Gamma \).

(ii) \( (t^s)^r = (t)^{sr} \) for all \( s, r \in S \) and \( t \in T \).

**Proof.** Let \( s, r \in S, \alpha \in \Gamma \) and \( t, u \in T \). Now \( (tau)^s = (\phi(s))(t)(au) = (\phi(s))(t)\alpha(\phi(s))(u) = t^s au^s \). Hence (i) follows. Again \( (t^s)^r = (\phi(r))(t^s) = (\phi(r))((\phi(s))(t)) = (\phi(r)\phi(s))(t) = (\phi(sr))(t) = (t)^{sr} \). Thus (ii) follows.

**Theorem 3.2.** Let \( S \times_\phi T \) be a semidirect product of a semigroup \( S \) and a \( \Gamma \)-semigroup \( T \). Then \( T^x \) is a \( \Gamma \)-semigroup for all \( x \in S \) where \( T^x = \{ t^x : t \in T \} \). If moreover \( S \times_\phi T \) is a regular \( \Gamma \)-semigroup then \( S \) is a regular semigroup and \( T^e \) is a regular \( \Gamma \)-semigroup for all \( e \in E(S) \).

**Proof.** The first part is clear from the above lemma. Let \( S \times_\phi T \) be regular. For \( (s, t) \in S \times_\phi T \), there exist \( (s', t') \in S \times_\phi T \) and \( \alpha, \beta \in \Gamma \) such that \( (s, t) = (s, t)\alpha(s', t')\beta(s, t) = (ss's, t^{s'}\alpha(t')^s\beta t) \) and \( (s', t') = (s', t')\beta(s, t)\alpha(s', t') = (ss's', (t')^{ss'}\beta t^{s'}\alpha t') \). This implies \( s' \in E(S) \). Let \( e \in E(S) \), then for \( (e, t^e) \), there exist \( (s', t') \in S \times_\phi T \) and \( \alpha, \beta \in \Gamma \) such that \( (e, t^e) = (e, t^e)\alpha(s', t')\beta(e, t^e) = (es'e, t^{es'}e\alpha t^e\beta t^e) \) and \( (s', t') = (s', t')\beta(e, t^e)\alpha(s', t') = (ss's', (t')^{es'}\beta t^{es'}\alpha t') \). Hence \( s' \in E(e) \) and we have \( t^e = t^{es'}e\beta t^e \) and \( t^e = t^e\beta t^e\alpha t^e \). i.e, \( t^e \in V''_{\alpha}(t^e) \). Hence \( T^e \) is a regular \( \Gamma \)-semigroup.
Theorem 3.3. Let $S$ be a semigroup and $T$ be a $\Gamma$-semigroup, $\phi : S \rightarrow \text{End}(T)$ be a given antimorphism. If the semidirect product $S \times_{\phi} T$ is an orthodox $\Gamma$-semigroup then $S$ is an orthodox semigroup and $T^e$ is a right (left) orthodox $\Gamma$-semigroup for every idempotent $e \in S$.

(i) a right (left) orthodox $\Gamma$-semigroup then $S$ is an orthodox semigroup and $T^e$ is a right (left) orthodox $\Gamma$-semigroup for every idempotent $e \in S$.

(ii) a right (left) inverse $\Gamma$-semigroup then $S$ is a right (left) inverse semigroup and $T^e$ is a right (left) inverse $\Gamma$-semigroup.

Proof.

(i) Let $S \times_{\phi} T$ be a right orthodox $\Gamma$-semigroup. Let $e, g \in E(S)$ and $t^e$ be an $\alpha$-idempotent and $u^e$ be a $\beta$-idempotent in $T^e$. Then $(e, t^e)\alpha(e, t^e) = (e, t^e\alpha t^e) = (e, t^e)$, i.e., $(e, t^e)$ is an $\alpha$-idempotent. Similarly $(e, u^e)$ is a $\beta$-idempotent. Again $(g, u^g)\beta(g, u^g) = (g, u^g\beta u^g) = (g, (u^e\beta u^g)^g) = (g, u^g)$. Thus $(g, u^g)$ is a $\beta$-idempotent of $S \times_{\phi} T$. Now $(e, (t^e\alpha u^e))\beta(t^e\alpha u^e) = (e, (t^e\alpha u^e))\beta(e, (t^e\alpha u^e)) = ((e, t^e)\alpha(e, u^e))\beta(e, (t^e)\alpha(e, u^e)) = (e, t^e)\alpha(e, u^e) = (e, t^e \alpha u^e)$ which shows that $t^e\alpha u^e$ is a $\beta$-idempotent and hence $T^e$ is a right orthodox $\Gamma$-semigroup. Again since $S \times_{\phi} T$ is a right orthodox $\Gamma$-semigroup we have $((eg)^2, (t^e g\alpha u^g)\beta t^e g\alpha u^g) = (eg, t^e g\alpha u^g)\beta(eg, t^e g\alpha u^g) = ((e, t^e)\alpha(g, u^g))\beta((e, t^e)\alpha(g, u^g)) = (e, t^e)\alpha(g, u^g) = (eg, t^e g\alpha u^g)$. Thus $(eg)^2 = eg$ which shows that $S$ is orthodox.

(ii) Suppose that $S \times_{\phi} T$ is a right inverse $\Gamma$-semigroup. Let $e, g \in E(S)$ and $u^e$ be a $\beta$-idempotent in $T^e$. Then $(e, t^e)$ is an $\alpha$-idempotent, $(e, u^e), (g, u^g)$ are $\beta$-idempotents of $S \times_{\phi} T$. Now $(e, t^e)\alpha(e, u^e)\beta(t^e) = (e, t^e)\alpha(e, u^e)\beta(e, t^e) = (e, u^e)\beta(e, t^e)$ and $(e, g)\alpha(g, u^g)\beta(t^e) = (e, g)\alpha(g, u^g)\beta(e, t^e) = (e, g)\beta(e, t^e) = (ge, u^g)\beta(t^e)$. So we have $t^e \alpha u^e \beta t^e = u^e \beta t^e$ and $ege = ge$. Consequently we have $S$ is a right inverse semigroup and $T^e$ is a right inverse $\Gamma$-semigroup.

The proofs of the following two theorems are almost similar to our Lemma 3.3 and Lemma 3.4 proved in [6]. For completeness we give the proof here.

Theorem 3.4. Let $S \times_{\phi} T$ be the semidirect product of a semigroup $S$ and a $\Gamma$-semigroup $T$ corresponding to a given antimorphism $\phi : S \rightarrow \text{End}(T)$ and let $(s, t) \in S \times_{\phi} T$, then
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(i) if $(s', t') \in V^\beta_\alpha((s, t))$ then $(s', t') \in V^\beta_\alpha((s, t^s s'))$. In particular if $s \in E(S)$, then $(s, (t')^s \beta t^s \alpha t') \in V^\beta_\alpha((s, t^s s'))$ and 

(ii) if $t^s$ is an $\alpha$-idempotent and $s' \in V(s)$, then $(s', t^{s s'} s') \in V^\alpha_\alpha((s, t^s))$.

**Proof.**

(i) Since $(s', t') \in V^\beta_\alpha((s, t))$ we have,

$$(s', t') = (s', t') \beta(s, t) \alpha(s', t') = (s' s s', (t')^{s s'} \beta t^s \alpha t')$$

and

$$(s, t) = (s, t) \alpha(s', t') \beta(s, t) = (s s s', t^{s s'} \alpha (t')^s \beta t).$$

This shows that

(1) $$s' \in V(s) \text{ and } t^{s s'} \alpha (t')^s \beta t = t$$

(2) $$(t')^{s s'} \beta t^s \alpha t' = t'.$$

From (1) we have, $(t')^{s s'} \alpha (t')^s \beta t^s = (t')^{s s'}$ i.e., $t^{s s'} \alpha (t')^s \beta t^{s s'} = t^{s s'}$ and from (2), $((t')^{s s'} \beta t^s \alpha t')^s = (t')^s$ i.e., $(t')^{s s'} \beta t^{s s'} \alpha (t')^s = (t')^s$. Now $(s', t') \beta(s, t^s s) \alpha(s', t') = (s' s s', (t')^{s s'} \beta t^{s s'} \alpha t') = (s', t')$ by (2) and $(s, t^s s) \alpha(s', t') \beta(s, t^s s) = (s s s, t^{s s s} \alpha(t')^s \beta t^{s s}) = (s, (t')^{s s} \alpha(t')^s \beta t^{s s}) = (s, t^{s s}).$ Thus we have $(s', t') \in V^\beta_\alpha((s, t^s s))$. Again if $s \in E(S)$, $((t')^s \beta t^s \alpha t')^s = (t')^{s s'} \beta t^{s s} \alpha (t')^s = (t')^s$ and $(s, t^s)^s \alpha(s, (t')^s \beta t^s s \alpha t')^s = (s, t^s s) \alpha(s, (t')^s \beta t^{s s} \alpha t')^s = (s, (t')^s \beta t^{s s} \alpha t')^s$ and $(s, (t')^s \beta t^{s s} \alpha t')^s = (s, (t')^s \beta t^s s \alpha t')^s = (s, (t')^s \beta t^{s s} \alpha t')^s$. Hence $(s, (t')^s \beta t^{s s} \alpha t')^s \in V^\beta_\alpha(s, t^{s s}).$

(ii) $(s, t^s)^s \alpha(s, t^{s s}) = (s s s, t^{s s s} \alpha t^{s s} \alpha t^s) = (s, t^s)$ since $t^s$ is an $\alpha$-idempotent and $(s, t^{s s}) \alpha(s, t^s) = (s s s, t^{s s s} \alpha t^{s s} \alpha t^s) = (s, t^{s s} \alpha t^{s s} \alpha t^s) = (s, (t^s \alpha t)^s \alpha t') = (s, t^{s s} \alpha t^s) i.e., (s, t^{s s}) \in V^\alpha_\alpha(s, t^s).

**Theorem 3.5.** Let $S$ be a semigroup and $T$ be a $\Gamma$-semigroup and $S \times \phi T$ be the semidirect product corresponding to a given antimorphism $\phi : S \rightarrow \text{End}(T)$. Moreover, if $t \in t^e \Gamma T$ for every $e \in E(S)$ and every $t \in T$, then

(i) $(e, t)$ is an $\alpha$-idempotent if and only if $e \in E(S)$ and $t^e$ is an $\alpha$-idempotent and 

(ii) if $(e, t)$ is an $\alpha$-idempotent, then $(e, t^e) \in V^\alpha_\alpha((e, t)).$
Proof.

(i) If \((e, t)\) is an \(\alpha\)-idempotent then

\[
(e, t) = (e, t)\alpha(e, t) = (e^2, t^e\alpha t) \text{ i.e., } e = e^2 \text{ and } t^e\alpha t = t.
\]

So, \(t^e = (t^e\alpha t)^e = t^e\alpha t^e\) which implies that \(t^e\) is an \(\alpha\)-idempotent. Conversely, let \(e \in E(S)\) and \(t^e\) be an \(\alpha\)-idempotent. Since \(t \in t^e\Gamma T\), \(t = t^e\beta t_1\) for some \(\beta \in \Gamma\), \(t_1 \in T\) and hence \(t^e\alpha t = t^e\alpha t^e\beta t_1 = t\). Thus \((e, t)\alpha(e, t) = (e, t^e\alpha t) = (e, t)\) i.e., \((e, t)\) is an \(\alpha\)-idempotent.

(ii) If \((e, t)\) is an \(\alpha\)-idempotent, from (i) \(e \in E(S)\) and \(t^e\) is an \(\alpha\)-idempotent.

Now \((e, t)\alpha(e, t^e)\alpha(e, t) = (e, t^e\alpha t^e\alpha t) = (e, t)\) from (3) and \((e, t^e)\alpha(e, t)\alpha(e, t^e) = (e, t^e\alpha t^e t^e) = (e, t^e)\). Thus \((e, t^e) \in V^\alpha_{\alpha}((e, t))\).

Theorem 3.6. Let \(S\) be a semigroup and \(T\) be a \(\Gamma\)-semigroup. Let \(\phi : S \not\rightarrow End(T)\) be a given antimorphism. Then the semidirect product \(S \times_\phi T\) is a right (left) orthodox \(\Gamma\)-semigroup if and only if

(i) \(S\) is an orthodox semigroup and \(T^e\) is a right (left) orthodox \(\Gamma\)-semigroup for every \(e \in E(S)\),

(ii) for every \(e \in E(S)\) and every \(t \in T\) \((\phi(t) = t^e)\) and

(iii) for every \(\alpha\)-idempotent \(t^e\), \(t^g^e\) is an \(\alpha\)-idempotent, where \(e, g \in E(S)\), \(t \in T\).

Proof. Suppose \(S \times_\phi T\) is a right orthodox \(\Gamma\)-semigroup. Then by Theorem 3.3 \(S\) is an orthodox semigroup and \(T^e\) is a right orthodox \(\Gamma\)-semigroup for every \(e \in E(S)\). For (ii), let \((e, t) \in S \times_\phi T\) with \(e \in E(S)\) and let \((e', t') \in V^\beta_{\alpha}((e, t))\) for some \(\alpha, \beta \in \Gamma\). Then by Theorem 3.4 \((e', t'), (e', t')^e \beta t^e\alpha t') \in V^\alpha_{\alpha}((e, t^e))\). Thus \(V^\alpha_{\alpha}((e, t)) \cap V^\beta_{\alpha}((e, t^e)) \neq \phi\) and hence by Theorem 2.9, \(V^\beta_{\alpha}((e, t)) = V^\alpha_{\alpha}((e, t^e))\). So \((e, t')^e \beta t^e\alpha t' \in V^\beta_{\alpha}((e, t))\). Thus \((e, t) = (e, t)\alpha(e, (t')^e \beta t^e\alpha t')\beta(e, t) = (e, t^e\alpha(t')^e \beta t^e\alpha(t')^e \beta t)\) and hence \(t = t^e\alpha(t')^e \beta t^e\alpha(t')^e \beta t \in t^e\Gamma T\).

For (iii) we shall first show that for an \(\alpha\)-idempotent \(t^e\) of \(T\) if \(e \in E(S), t^e\) is an \(\alpha\)-idempotent for any \(e' \in V(e)\). If \(e \in E(S)\) and \(t^e\) is an \(\alpha\)-idempotent, then by Theorem 3.5, \((e, t)\) is an \(\alpha\)-idempotent in \(S \times_\phi T\) and \((e, t^e) \in V^\alpha_{\alpha}((e, t))\). Again since \(t^e\) is an \(\alpha\)-idempotent \((e, t^e)\) is also an \(\alpha\)-idempotent and thus \(e, t^e) \in V^\alpha_{\alpha}((e, t^e))\) i.e., \(V^\alpha_{\alpha}((e, t^e)) \cap V^\alpha_{\alpha}((e, t)) \neq \phi\) and so \(V^\alpha_{\alpha}((e, t^e)) = V^\alpha_{\alpha}((e, t))\) and by Theorem 3.5 \((e', t^e') \in V^\alpha_{\alpha}((e, t^e))\) i.e.,
(e', t^{ee'}) \in V^\alpha_0((e, t)). Thus (e, t) = (e, t)a(e', t^{ee'})a(e, t) = (ee'e, t^{ee'}at^{ee'}e at) = (e, t^{ee'}at^{ee}at) = (e, t^{ee}at) [since t = t^{ee}u for some \beta \in \Gamma, u \in T, t^{ee}at = t].

So t = t^{ee}at and hence \(t^{ee'} = (t^{ee}at)^{ee'} = t^{ee'}at^{ee'}\). Thus \(t^{ee'}\) is an \(\alpha\)-idempotent.

Let \(e, g \in E(S)\) and suppose that \(t^{ee}\) is an \(\alpha\)-idempotent for \(t \in T\), then \(t^{ee}at^{ee} = (t^{ee}at)^{ee} = t^{ee}\) i.e., \(t^{ee}\) is an \(\alpha\)-idempotent and we have \(eg \in E(S)\) and \(ge \in V(eg)\) since \(S\) is orthodox. Then by the above fact \(t^{ee}\) is an \(\alpha\)-idempotent.

We now prove the converse part. Suppose \(S\) and \(T\) satisfy (i), (ii) and (iii). Let \((s, t) \in S \times \phi T\). Since \(S\) is regular, there exists \(s' \in S\) such that \(s = ss'ss\). We take \(e = s's\), then \(e \in E(S)\). By (ii) \(t \in t^\Gamma T\) which implies \(t = t^{ee}at^{ee}at\) for some \(\beta \in \Gamma, u \in T\). Let \(t' = v^{ss}s\) where \(v^{ss} \in V^\alpha_2(t^{ee})\). Now \(t^{ee}sata = t^{ee}a\) for every \(s \in S\), \(a \in A\). Again \((s')^{ee}a = (s')^{ee}a\) for every \(s \in S\), \(a \in A\). We have \((s', t') \in V^\alpha_2((s, t))\) which yields \(S \times \phi T\) is a regular \(\Gamma\)-semigroup.

Now let \((e, t)\) be an \(\alpha\)-idempotent and \((g, u)\) be a \(\beta\)-idempotent. Then by Theorem 3.5 \(e, g \in E(S), t^{ee}\) is an \(\alpha\)-idempotent and \(u^{ee}\) is an \(\beta\)-idempotent. By (iii) \(t^{ee}\) is an \(\alpha\)-idempotent, \(u^{ee}\) is a \(\beta\)-idempotent and \(t^{ee}at^{ee}at = (t^{ee}a)^{ee}at = t^{ee}\) i.e., \(t^{ee}\) is an \(\alpha\)-idempotent. By our assumption \(e, g \in E(S)\) and \((t^{ee}a)^{ee}at = t^{ee}a\) is a \(\beta\)-idempotent. Thus by Theorem 3.5 \((e, t)a(e, u) = (eg, t^{ee}a)\) is a \(\beta\)-idempotent which shows that \(S \times \phi T\) is a right orthodox \(\Gamma\)-semigroup.

**Theorem 3.7.** Let \(S \times \phi T\) be a right inverse \(\Gamma\)-semigroup. Then by Theorem 3.3 \(S\) is a right inverse \(\Gamma\)-semigroup and \(S \times \phi T\) is a right inverse \(\Gamma\)-semigroup if and only if

(i) \(S\) is a right inverse semigroup and \(T^e\) is a right inverse \(\Gamma\)-semigroup for every \(e \in E(S)\) and

(ii) for every \(e \in E(S)\) and every \(t \in T, t \in t^\Gamma T\).

**Proof.** Let \(S \times \phi T\) be a right inverse \(\Gamma\)-semigroup. Then by Theorem 3.3 \(S\) is a right inverse \(\Gamma\)-semigroup and \(T^e\) is a right inverse \(\Gamma\)-semigroup for every \(e \in E(S)\). Again since every right inverse \(\Gamma\)-semigroup is a right orthodox \(\Gamma\)-semigroup from the above theorem, condition (ii) holds.

Conversely, suppose that \(S\) and \(T\) satisfy (i) and (ii). Then by Theorem 3.2 \(S \times \phi T\) is a regular \(\Gamma\)-semigroup. Let \((e, t)\) be an \(\alpha\)-idempotent and \((g, u)\)
be a $\beta$-idempotent in $S \times \phi T$. Then by Theorem 3.5, $e, g \in E(S)$, $t^e$ is an $\alpha$-idempotent, $u^g$ is a $\beta$-idempotent. From (ii) $t = t^e \gamma v$ for some $\gamma \in \Gamma$, $v \in T$ and thus $t^e a t = t$ and similarly $u^g \beta u = u$. So $u^g = (u^g \beta u)^g = u^g \beta u^g$ and $t^g = (t^e a t)^g = t^e \gamma v^g \alpha t^g = t^g \alpha t^g$ since $S$ is a right inverse semigroup. Now by (ii) we have $u^e \beta t = (u^e \beta t)^{ge} \delta v$ for some $\delta \in \Gamma$, $v \in T$ and hence $u^e \beta t = u^{ge} \beta t^{ge} \delta \delta v$. Thus we have $(e, t) \alpha (g, u) \beta (e, t) = (e g e, t^g \alpha u^g \beta t)$ = $(g e, u^g \beta t^{ge} \delta v_1) = (g e, u^g \beta t^{ge} \delta v_1) = (g e, u^g \beta t^{ge} \delta v_1)$ which implies $S \times \phi T$ is a right inverse $\Gamma$-semigroup.

**Theorem 3.8.** Let $S$ be a semigroup, $T$ be a $\Gamma$-semigroup and $\phi : S \not\rightarrow \text{End}(T)$ be a given antimorphism. Then the semidirect product $S \times \phi T$ is a left inverse $\Gamma$-semigroup if and only if

(i) $S$ is a left inverse semigroup and $T^e$ is a left inverse $\Gamma$-semigroup for every $e \in E(S)$ and

(ii) for every $e \in E(S)$ and every $t \in T$, $t = t^e$.

**Proof.** Let $S \times \phi T$ be a left inverse $\Gamma$-semigroup. Then by Theorem 3.3 $S$ is a left inverse semigroup and $T^e$ is a left inverse $\Gamma$-semigroup. For (ii) let $(e, u)$ be an $\alpha$-idempotent in $S \times \phi T$. Then $(e, u) = (e, u) \alpha (e, u) = (e, u^e \alpha u)$ i.e., $u^e \alpha u = u$. Again $(e, u^e) \alpha (e, u^e) = (e, u^e \alpha u^e)$ which yields $(e, u^e)$ is an $\alpha$-idempotent and we have $(e, u^e) \alpha (e, u) = (e, u^e \alpha u) = (e, u)$. Since $S \times \phi T$ is a left inverse $\Gamma$-semigroup, $(e, u) \alpha (e, u) = (e, u^e) \alpha (e, u) = (e, u^e \alpha u)$, $(e, u) \alpha (e, u) = (e, u^e \alpha u)$ i.e., $u = u^e$. Thus if $(e, u)$ is an $\alpha$-idempotent then $u = u^e$. Now $(e, t) \in S \times \phi T$ with $e \in E(S)$ and let $(e', t') \in V_\gamma((e, t))$ for some $\gamma, \delta \in \Gamma$. Then we get $e' \in V(e)$, $t^e \gamma(t') \delta t = t$ i.e., $t^e \gamma(t') \delta t^{e'} = t^{e'}$ which implies $t^e \gamma(t') \delta t^{e'} = t^{e'}$. Since $(e', t') \delta t = (e', t') \delta t$ and $S \times \phi T$ is left orthodox (since it is left inverse), $(e', t') \delta t$ is a $\gamma$-idempotent and hence $(t') \delta t = ((t') \delta t)^{e'} = (t') \delta t^{e'}$. Thus $t^e = t^{e'} \gamma(t') \delta t^{e'} = t^{e'} \gamma(t') \delta t = t$ and hence $t^e = (t^{e'})^e = t^{e'} = t$.

Conversely suppose that $S$ and $T$ satisfy (i) and (ii). Let $(s, t) \in S \times \phi T$. Let $e \in E(S)$. Since $S$ is regular there exists $s' \in S$ such that $s' \in V(s)$. From (ii) we have $t = t^e$. Since $T^e$ is regular there exists $v \in T$ such that $v^e \in V_\gamma'(t^e)$. We now take $t^e = v^{s'}$. Now $t^{s'} \gamma(t') \delta t = t^{s'} \gamma v^{s'} \delta t = t^{s'} \gamma v^{s'} \delta t = t^e = t$ and $(t')^{s'} \delta t^{s'} \gamma t' = (v^{s'})^{s'} \delta t^{s'} \gamma v^{s'} = v^{s'} \delta t^{s'} \gamma v^{s'} = v^{s'} \delta t^{s'} \gamma v^{s'} = (v^{s'} \delta t^{s'} \gamma v^{s'})^{s'} = v^{s'} = v^{s'} = v^t$. Thus we have $(s', t') \in V_\gamma'(s, t)$. Hence $S \times \phi T$ is regular. Now let $(e, t)$ be an $\alpha$-idempotent and $(g, u)$ be a
β-idempotent. Then \( e^2 = e \) and \( t = t^αa = tat \) [by (ii)] and similarly \( g^2 = g \)
and \( uβu = u \) i.e., \( e, g \in E(S) \) and \( t \) is an \( α \)-idempotent, \( u \) is a β-idempotent.
Thus we have \((e, t)β(g, u)α(e, t) = (ege, t^αβu^αat) = (ege, tβuat) \) [by (ii)]
\( = (eg, tβu) = (eg, t^βu) = (e, t)β(g, u) \). Thus \( S \times_φ T \) is a left inverse Γ-
semigroup.

4. Wreath product of a semigroup and a Γ-semigroup

In this section we introduce the notion of wreath product of a semigroup \( S \) and a Γ-semigroup \( T \). Let \( X \) be a nonempty set. Consider the set \( T^X \) of
all mappings from \( X \) to \( T \). For \( f, g \in T^X \) and \( α \in Γ \), define \( fαg \) such that
\( T^X \times Γ \times T^X \rightarrow T^X \) by \((fαg)(x) = f(x)αg(x) \).

Before going to establish the relation between \( T \) and \( T^X \) we assume
\( Γ = \{α\} \), a set consisting of single element. Then \((T, \cdot) \) becomes a semigroup
where \( a \cdot b = aab \) and \( T^X \) also becomes a semigroup where \( f \cdot g = fαg \).
Suppose \( T \) is a regular Γ-semigroup. Then \((T, \cdot) \) is a semigroup. Let \( f \in T^X \)
and let \( x \in X \). Now \( f(x) \in T \) and \( V(f(x)) \neq φ \). We define \( g : X \rightarrow T \)
so that \( g(x) \in V(f(x)) \). Hence for each \( x \in X \) we can choose a \( g(x) \) such
that \( f(x)g(x)f(x) = f(x) \). Hence \( fgf = f \) which implies that \((T^X, \cdot) \) is a regular semigroup and consequently \( T^X \) is a regular Γ-semigroup. In general
we cannot extend the process when \( Γ \) contains more than one element. To
explain this we consider the following example.

**Example 4.1.** Let \( T = \{(a, 0) : a \in Q\} \cup \{(0, b) : b \in Q\} \), \( Q \) denote the
set of all rational numbers. Let \( Γ = \{(0, 5), (0, 1), (3, 0), (1, 0)\} \). Defining
\( T \times Γ \times T \rightarrow T \) by \((a, b)(α, β)(c, d) = (aac, bβd) \) for all \((a, b), (c, d) \in T \)
and \((α, β) \in Γ \), we can show that \( T \) is a Γ-semigroup. Now let \((a, 0) \in T \). If \( a = 0 \)
then \((a, 0) \) is regular. Suppose \( a \neq 0 \), then \((a, 0)(3, 0)(\frac{1}{3a}, 0)(1, 0)(a, 0) = (a, 0) \).
Similarly we can show that \((0, b) \) is also regular. Hence \( T \) is a regular Γ-
semigroup. Let us now take a set \( X = \{x, y\} \), the set consisting of two
elements and let us define a mapping \( f : X \rightarrow T \) by \( f(x) = (2, 0) \) and
\( f(y) = (0, 3) \). We now show that \( f \) is not regular in \( T^X \). If possible let \( f \)
be regular. Then there exists a mapping \( g : X \rightarrow T \) and two elements
\( α, β \in Γ \) such that \( fαgβf = f \). i.e., \( f(p)αg(p)βf(p) = f(p) \) for all \( p \in X \).
Now if \( p = x \), then \( α, β \notin \{(0, 5), (0, 1)\} \), since the first component of \( f(x) \) is
nonzero but if \( p = y \), then \( α, β \notin \{(0, 5), (0, 1)\} \), since the second component
of \( f(y) \) is nonzero. Thus a contradiction arises. Hence \( T^X \) is not a regular Γ-
semigroup.
Before further discussion about the relation between $T$ and $T^X$ we now give the following definition.

**Definition 4.2.** Let $S$ be a $\Gamma$-semigroup. An element $e \in S$ is said to be a left (resp. right) $\gamma$-unity for some $\gamma \in \Gamma$ if $e\gamma a = a$ (resp. $a\gamma e = a$) for all $a \in S$.

We now consider the following examples.

**Example 4.3.** Consider the $\Gamma$-semigroup $S$ of Example 2.3. In this $\Gamma$-semigroup \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] is a left $\alpha$-unity but not a right $\alpha$-unity of $S$ for $\alpha = \begin{pmatrix} 1 & 0 \\
0 & 1 \\
0 & 0 \end{pmatrix}$.

**Example 4.4.** Let $S$ be the set of all integers of the form $4n+1$ and $\Gamma$ be the set of all integers of the form $4n+3$ where $n$ is an integer. If $aob$ is $a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$ then $S$ is a $\Gamma$-semigroup. Here $1$ is a left $(-1)$-unity and also right $(-1)$- unity.

**Example 4.5.** Let us consider $N$, the set of all natural numbers. Let $S$ be the set of all mappings from $N$ to $N \times N$ and $\Gamma$ be the set of all mappings from $N \times N$ to $N$. Then the usual mapping product of two elements of $S$ cannot be defined. But if we take $f, g$ from $S$ and $\alpha$ from $\Gamma$ the usual mapping product $f\alpha g$ can be defined. Also, we find that $f\alpha g \in S$ and $(f\alpha g)\beta h = f\alpha(g\beta h)$. Hence $S$ is a $\Gamma$-semigroup. Now we know that the set $N \times N$ is countable. Hence there exists a bijective mapping $f \in S$. Since $f$ is bijective, there exists $\alpha : N \times N \rightarrow N$ such that $f\alpha$ is the identity mapping on $N \times N$ and $\alpha f$ is the identity mapping on $N$. Then $f\alpha g = g\alpha f = g$ for all $g \in S$. Hence $f$ is both left $\alpha$-unity and right $\alpha$-unity of $S$.

Let $S$ be a $\Gamma$-semigroup and $e$ be a left $\alpha$-unity. Then $STe$ is a left ideal such that $e = eae \in STe$. Also we note that the element $e$ is both left and right $\alpha$-unity of $STe$ in $STe$.

Suppose $S$ is a regular $\Gamma$-semigroup with a left $\alpha$-unity $e$. Then we show that $STe$ is a regular $\Gamma$-semigroup with a unity. We only show that $STe$ is regular. Let $a\gamma e \in STe$. Since $S$ is regular there exist $\beta, \delta \in \Gamma$ and $b \in S$ such that $a\gamma e = a\gamma e\beta\delta\alpha \gamma e$ i.e., $a\gamma e = a\gamma e\beta\delta\alpha \gamma e = (a\gamma e)\beta(\delta e)\alpha(a\gamma e)$. Since $\delta e \in STe$, $a\gamma e$ is regular. Hence $STe$ is a regular $\Gamma$-semigroup.
Let us now consider $T$ with a left $\gamma$-unity $e$ and a right $\delta$-unity $g$. Then the constant mapping $C_e : X \to T$ which is defined by $C_e(x) = e$ for all $x \in X$ is a left $\gamma$-unity of $T^X$. Similarly the constant mapping $C_g$ is a right $\delta$-unity of $T^X$.

**Theorem 4.6.** Let $T$ be a $\Gamma$-semigroup with a left $\gamma$-unity and a right $\delta$-unity for some $\gamma, \delta \in \Gamma$. Then

(i) $T^X$ is a regular $\Gamma$-semigroup if and only if $T$ is a regular $\Gamma$-semigroup,

(ii) $T^X$ is a right (resp. left) orthodox $\Gamma$-semigroup if and only if $T$ is so and

(iii) $T^X$ is a right (resp. left) inverse $\Gamma$-semigroup if and only if $T$ is a right (resp. left) inverse $\Gamma$-semigroup.

**Proof.** By $C_t, t \in T$ denotes the mapping in $T^X$ such that $C_t(x) = t$ for all $x \in X$. Then it is clear that $(C_t)\alpha(C_u) = C_{(tu)}$ which shows that $C_t$ is an $\alpha$-idempotent if and only if $t$ is an $\alpha$-idempotent. Again we have that if $f$ is an $\alpha$-idempotent in $T^X$ then $f(x)$ is an $\alpha$-idempotent in $T$ for all $x \in X$.

(i) Assume that $T^X$ is a regular $\Gamma$-semigroup. Then for each $t \in T$ there exist $f \in T^X$ and $\alpha, \beta \in \Gamma$ such that $C_t\alpha f\beta C_t = C_t$ so that $t\alpha f(x)\beta t = t$ for all $x \in X$ which shows that $t$ is regular in $T$. Consequently $T$ is a regular $\Gamma$-semigroup. Conversely let $T$ be regular and let $e$ be a left $\gamma$-unity and $g$ be a right $\delta$-unity of $T$. Then for each $f \in T^X$ and for each $x \in X$, $f(x) \in T$ is a regular element and hence there exists a triplet $(\alpha_x, t_x, \beta_x) \in \Gamma \times T \times \Gamma$ such that $f(x)\alpha_x t_x \beta_x f(x) = f(x)$. i.e., $f(x) = (f(x)\delta y)\alpha_x t_x \beta_x (e\gamma f(x)) = f(x)\delta (go_x t_x \beta_x e)\gamma f(x)$. Define $h : X \to T$ by $h(x) = go_x t_x \beta_x e$. Then for all $y \in X$, we have

$$(f\delta h \gamma f)(y) = f(y)\delta h(y)\gamma f(y)$$

$$= f(y)\delta g\alpha_y t_y \beta_y e\gamma f(y)$$

$$= f(y)\alpha_y t_y \beta_y f(y)$$

$$= f(y).$$

Hence $f$ is regular in $T^X$. Consequently $T^X$ is a regular $\Gamma$-semigroup.
(ii) Let \( t, u \in T \) such that \( t \) be an \( \alpha \)-idempotent and \( u \) be a \( \beta \)-idempotent.

Then \( C_t \) is an \( \alpha \)-idempotent and \( C_u \) is a \( \beta \)-idempotent in \( T^X \). Now if \( T^X \) is a right orthodox \( \Gamma \)-semigroup then \((C_t \alpha C_u) \beta (C_t \alpha C_u) = C_t \alpha C_u\)
i.e., \( t \alpha u \) is a \( \beta \)-idempotent in \( T \) which implies \( T \) is also a right orthodox \( \Gamma \)-semigroup. Similarly we can show that if \( T^X \) is a left orthodox \( \Gamma \)-semigroup then \( T \) is so. Let \( f \) be an \( \alpha \)-idempotent and \( h \) be a \( \beta \)-idempotent in \( T^X \). Let us now suppose that \( T \) is a right (resp. left) orthodox \( \Gamma \)-semigroup. Then \( f(x) ah(x) \beta h(x) \) is a \( \beta \)-idempotent (resp. \( \alpha \)-idempotent). Hence \( T^X \) is a right (resp. left) orthodox \( \Gamma \)-semigroup.

(iii) Let \( T^X \) be a right (resp. left) inverse \( \Gamma \)-semigroup and let \( t, u \in T \) such that \( t \) is an \( \alpha \)-idempotent and \( u \) be a \( \beta \)-idempotent. Then \( C_t \) is an \( \alpha \)-idempotent and \( C_u \) is a \( \beta \)-idempotent in \( T^X \) and \( C_t \alpha C_u \beta C_t = C_u \beta C_t \beta (C_t \beta C_u) \). Thus we have \( t \alpha u \beta t = u \beta t \beta \alpha t = t \beta u \alpha t \) which implies that \( T \) is a right (resp. left) inverse \( \Gamma \)-semigroup. Again let \( T \) be a right (resp. left) inverse \( \Gamma \)-semigroup. Let \( f \) be an \( \alpha \)-idempotent and \( h \) be a \( \beta \)-idempotent in \( T^X \). \( f(x) ah(x) \beta f(x) = h(x) \beta f(x) \) (resp. \( f(x) \beta h(x) \alpha f(x) = f(x) \beta h(x) \)) for all \( x \in X \) i.e., \( fah \beta f = h \beta f \) (resp. \( f \beta haf = f \beta h \)). Thus \( T^X \) is a right (resp. left) inverse \( \Gamma \)-semigroup.

Let us now suppose that the semigroup \( S \) acts on \( X \) from the left i.e.,
\[ sx \in X, s(rx) = (sr)x \] and \( lx = x \) if \( S \) is a monoid, for every \( r, s \in S \) and every \( x \in X \). If \( S \) acts on \( X \) from left we call it left \( S \) set \( X \).

For every \( \Gamma \)-semigroup \( T \), it is known that \( \text{End}(T) \) is a semigroup. Hence \( \text{End}(T^X) \) is also a semigroup.

Let \( S \) be a semigroup, \( T \) a \( \Gamma \)-semigroup and \( X \) a nonempty set. Suppose \( S \) acts on \( X \) from left. Define \( \phi : S \to \text{End}(T^X) \) by \( ((\phi(s))f)(x) = f(sx) \) for all \( s \in S, f \in T^X \) and \( x \in X \). We now verify that \( \phi(s) \in \text{End}(T^X) \). For this, let \( f, g \in T^X, \alpha \in \Gamma \) and \( x \in X \). Then \( ((\phi(s))((f \circ g))(x) = (f \circ g)(sx) = f(sx) \alpha g(sx) = ((\phi(s))(f))(x) \alpha ((\phi(s))(g))(x) = ((\phi(s))(f))(x) \alpha ((\phi(s))(g))(x) \). Hence \( (\phi(s))(f)\circ g = \alpha ((\phi(s))(g)) \), which implies that \( \phi(s) \in \text{End}(T^X) \).

Let us now verify that \( \phi : S \to \text{End}(T^X) \) is a semigroup antimorphism. For this let \( s_1, s_2 \in S, f \in T^X \) and \( x \in X \). Then \( ((\phi(s_1) \phi(s_2))(f))(x) = (\phi(s_1))(\phi(s_2))(f))(x) = (\phi(s_2)(f))(s_1)(x) = f((s_2)(s_1)) = f((s_1)(s_2)) = (\phi(s_1)(s_2))(f))(x) \). Hence \( \phi(s_1 s_2) = \phi(s_1) \phi(s_2) \).
For this antimorphism \( \phi : S \not\rightarrow \text{End}(T^X) \) we can define the semidirect product \( S \times \phi T^X \) of the semigroup \( S \) and the \( \Gamma \)-semigroup \( T^X \). We call this semidirect product the wreath product of the semigroup \( S \) and the \( \Gamma \)-semigroup \( T \) relative to the left \( S \)-set \( X \). We denote it by \( S \wr_X T \). We also denote \( \phi(s)(f)(x) \) by \( f^s(x) \). Hence \( f^s(x) = f(sx) \).

If \( |T| = 1 \), then \( |T^X| = 1 \) and hence throughout the paper we assume that \( |T| \geq 2 \). We now give the relation between \( T \) and \( (T^X)^e \) for all \( e \in E(S) \).

Similar to the Theorems 3.6 and 3.7 we have following Theorems.

**Theorem 4.7.** Let \( S \) be a semigroup acting on the set \( X \) from the left and \( T \) be a \( \Gamma \)-semigroup with a left \( \gamma \)-unity and a right \( \delta \)-unity for some \( \gamma, \delta \in \Gamma \). Then

(i) \( T \) is a regular \( \Gamma \)-semigroup if and only if \( (T^X)^e \) is a regular \( \Gamma \)-semigroup,

(ii) \( T \) is a right (resp. left) orthodox \( \Gamma \)-semigroup if and only if \( (T^X)^e \) is so and

(iii) \( T \) is a right (resp. left) inverse \( \Gamma \)-semigroup if and only if \( (T^X)^e \) is a right (resp. left ) inverse \( \Gamma \)-semigroup.

**Theorem 4.8.** Let \( S \) be a semigroup acting on the set \( X \) from the left and \( T \) be a \( \Gamma \)-semigroup with a left \( \gamma \)-unity and a right \( \delta \)-unity for some \( \gamma, \delta \in \Gamma \). Then the wreath product \( S \wr_X T \) is a right(left) orthodox \( \Gamma \)-semigroup if and only if

(i) \( S \) is an orthodox semigroup and \( (T^X)^e \) is a right(left) orthodox \( \Gamma \)-semigroup for every \( e \in E(S) \)

(ii) for every \( x \in X, f \in T^X \) and \( e \in E(S), f(x) \in f(ex)T \) and

(iii) \( f(ex) \) is an \( \alpha \)-idempotent for every \( x \in X \), implies that \( f(gex) \) is an \( \alpha \)-idempotent for every \( g \in E(S) \) where \( e \in E(S), f \in T^X \).

We now prove the following Theorem.

**Theorem 4.9.** Let \( S \) be an orthodox semigroup acting on the set \( X \) from the left and \( T \) be a right orthodox \( \Gamma \)-semigroup with a left \( \gamma \)-unity and a right \( \delta \)-unity for some \( \gamma, \delta \in \Gamma \). Then the following statements are equivalent.

(a) \( S \) and \( T^X \) satisfy (ii) and (iii) of Theorem 4.8.

(b) \( S \) permutes \( X \) or \( T \) is a \( \Gamma \)- group and \( g \in X \subseteq eX \) for every \( e, g \in E(S) \).
Proof. (a) \implies (b): Let us suppose that \( T \) is not a \( \Gamma \)-group. Then there exists \( z \in T \) such that \( z\Gamma T \neq T \). Let \( e_\delta \) be a left \( \delta \)-unity in \( T \). For \( x \in X \), define \( f_x : X \to T \) by \( f_x(y) = e_\delta \) if \( y = x \) and \( f_x(y) = z \) if \( y \neq x \). Then by (ii), \( e_\delta = f_x(x) \in f_x(gx)\Gamma T \) for every \( g \in E(S) \). If \( f_x(gx) = z \) then \( e_\delta \in z\Gamma T \). Thus \( e_\delta = zov \) for some \( v \in T \) and \( \alpha \in \Gamma \). This implies that \( u = e_\delta \delta u = zov\delta u \) for all \( u \in T \). Hence \( T = z\Gamma T \) which is a contradiction.

Hence \( f_x(gx) = e_\delta \) Thus we can conclude that \( gx = x \) for all \( g \in E(S) \).

Let \( a \in S \) and \( x, y \in X \) such that \( ax = ay \). For \( a' \in V(a), a'a \in E(S) \) and \( x = (a'a)x = (a'a)y = y \). Again \( (aa')x = x \) implies that \( a(a'x) = x \).

Hence for each \( a \in S \), the mapping \( f_a : X \to X \) defined by \( f_a(x) = ax \) is a permutation on \( X \). This means that \( S \) permutes \( X \).

Now \( T \) is a \( \Gamma \)-group. Note that \( e_\delta \) is a \( \delta \)-idempotent and since \( T \) is a \( \Gamma \)-group, \( E_\delta(T) = \{e_\delta\} \). Let \( t \neq e_\delta \in T \) and \( e \in E(S) \). Define \( h : X \to T \) by \( h(x) = e_\delta \) if \( x \in eX \), otherwise \( h(x) = t \). Now \( h(ex) = e_\delta \) for every \( x \in X \) and hence by (iii), \( h(gex) = e_\delta \). This implies that \( gex \in eX \) and hence \( geX \subseteq eX \) for all \( e, g \in E(S) \).

(b) \implies (a): The proof is almost similar to the proof of (2) \implies (1) of Lemma 3.2 [5].

From Theorem 4.7 and 4.9 we conclude that

**Theorem 4.10.** Let \( S \) be a semigroup acting on the set \( X \) from the left and \( T \) be a \( \Gamma \)-semigroup with a left \( \gamma \)-unity and a right \( \delta \)-unity for some \( \gamma, \delta \in \Gamma \). Then the wreath product \( SW_XT \) is a right orthodox \( \Gamma \)-semigroup if and only if

1. \( S \) is an orthodox semigroup and \( T \) is a right orthodox \( \Gamma \)-semigroup and
2. \( S \) permutes \( X \) or \( T \) is a \( \Gamma \)-group and \( geX \subseteq eX \) for every \( e, g \in E(S) \).

**Theorem 4.11.** Let \( S, T \) and \( X \) be as in Theorem 4.10. Then the wreath product \( SW_XT \) is a right inverse \( \Gamma \)-semigroup if and only if

1. \( S \) is a right inverse semigroup and \( T \) is a right inverse \( \Gamma \)-semigroup and
2. \( S \) permutes \( X \) or \( T \) is a \( \Gamma \)-group.

Proof. Suppose that \( SW_XT \) is a right inverse \( \Gamma \)-semigroup. Then by Theorem 3.7 and Theorem 4.7 we have \( S \) is a right inverse semigroup and \( T \) is a right inverse \( \Gamma \)-semigroup and by Theorem 4.10 we have \( S \) permutes \( X \) or \( T \) is a \( \Gamma \)-group.
Conversely suppose that $S, T$ and $X$ satisfy (i) and (ii). Then by Theorem 4.6 $T^X$ is a right inverse $\Gamma$-semigroup. If $T$ is a $\Gamma$-group, then $f(x) \in f(ex)\Gamma T$ for every $f \in T^X$, $e \in E(S), x \in X$. If $S$ permutes $X$, then $f(x) \in f(x)\Gamma T = f(ex)\Gamma T$ since $ex = x$ for every $e \in E(S)$. Then by Theorem 3.7 $S \times_a T^X = SW_XT$ is a right inverse $\Gamma$-semigroup.

**Theorem 4.12.** Let $S, T$ and $X$ be as in Theorem 4.10. Then the wreath product $SW_XT$ is a left inverse $\Gamma$-semigroup if and only if $S$ is a left inverse semigroup and $T$ is a left inverse $\Gamma$-semigroup and $S$ permutes $X$.

**Proof.** By Theorem 3.8 and Theorem 4.7, we have $SW_XT$ is a left inverse $\Gamma$-semigroup if and only if $S$ is a left inverse semigroup and $T$ is a left inverse $\Gamma$-semigroup and $f(ex) = f(x)$ for every $f \in T^X$, $e \in E(S), x \in X$. The remaining part of the proof is almost similar to the proof of Corollary 3.7 [5].

**Open problem:**

(i) Find relation between $T$ and $T^X$ without assuming the existence of left $\alpha$-unity and right $\beta$-unity in the $\Gamma$-semigroup $T$ for some $\alpha, \beta \in \Gamma$.

(ii) Study the Wreath product of a semigroup $S$ and a $\Gamma$-semigroup $T$ without assuming the existence of left $\alpha$-unity and right $\beta$-unity in $T$ for some $\alpha, \beta \in \Gamma$.

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**References**


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