EXISTENCE RESULTS FOR DELAY SECOND ORDER DIFFERENTIAL INCLUSIONS

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Abstract

In this paper, some fixed point principle is applied to prove the existence of solutions for delay second order differential inclusions with three-point boundary conditions in the context of a separable Banach space. A topological property of the solutions set is also established.

Keywords: boundary-value problems, delay differential inclusions, fixed point, retract.

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1. Introduction, notation and preliminaries

Let $(E, \| \cdot \|)$ be a separable Banach space with a topological dual $E'$. $B(0, \rho)$ is the closed ball of $E$ of center $0$ and radius $\rho > 0$. By $L^1_E([0,1])$ we denote the space of all Lebesgue-Bochner integrable $E$-valued functions defined on $[0,1]$. Let $C_E([0,1])$ be the space of all continuous mappings $u : [0,1] \to E$, endowed with the sup norm.

Recall that a mapping $v : [0,1] \to E$ is said to be scalarly derivable when there exists some mapping $\hat{v} : [0,1] \to E$ (called the weak derivative of $v$) such that, for every $x' \in E'$, the scalar function $\langle x', v(\cdot) \rangle$ is derivable and its derivative is equal to $\langle x', \hat{v}(\cdot) \rangle$. The weak derivative $\hat{v}$ of $v$ when it exists is the weak second derivative.

By $W^{1,1}_E([0,1])$ we denote the space of all continuous mappings $u \in C_E([0,1])$ such that their first usual derivatives $\dot{u}$ are continuous and scalarly derivable and such that $\ddot{u} \in L^1_E([0,1])$. 
For closed subsets $A$ and $B$ of $E$, the Hausdorff distance $\mathcal{H}(A, B)$ between $A$ and $B$ is defined by

$$\mathcal{H}(A, B) = \max \left[ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right],$$

where

$$d(a, B) = \inf_{b \in B} \|a - b\|.$$

Let $r > 0$ and $\theta$ be a given number in $[0, 1]$. The aim of our paper is to provide existence of solutions for the second order delay-differential inclusion:

$$\begin{align*}
\mathcal{P}_r & \quad \left\{ \begin{array}{l}
\ddot{u}(t) \in F(t, u(t), u(h(t)), \dot{u}(t)), \text{ a.e. } t \in [0, 1] \\
u(t) = \varphi(t), \quad \forall t \in [-r, 0] \\
u(0) = 0; \quad u(\theta) = u(1).
\end{array} \right.
\end{align*}$$

We consider $F : [0, 1] \times E \times E \times E \rightrightarrows E$, $h : [0, 1] \to [-r, 1]$, $t - r \leq h(t) \leq t$, and $\varphi : [-r, 0] \to E$. The given mappings $h$ and $\varphi$ are continuous and $F$ is a convex compact valued multifunction Lebesgue-measurable on $[0, 1]$ and upper semi-continuous on $E \times E \times E$.

A solution $u$ of $(\mathcal{P}_r)$ is a mapping $u : [-r, 1] \to E$ satisfying $\ddot{u}(t) \in F(t, u(t), u(h(t)), \dot{u}(t))$ for almost every $t \in [0, 1]$, $u(t) = \varphi(t)$, for all $t \in [-r, 0]$ and $u(0) = 0; u(\theta) = u(1)$, with $u \in X := C_E([-r, 1]) \cap W^{2,1}_E([0, 1])$ equipped with the norm

$$\|u\|_X = \max \left\{ \sup_{t \in [-r, 1]} \|u(t)\|, \sup_{t \in [0, 1]} \|\dot{u}(t)\| \right\}.$$ 

In the second order evolution inclusions some related results are given in [1, 12, 15, 16, 17] and [18].

The existence of solutions for the second order delay differential problems have been discussed in the literature. For example, the problem described by the delay differential equation

$$\ddot{u}(t) = f(t, u(t), u(h(t)), \dot{u}(t)), \quad t \in [0, T]$$

with the boundary conditions

$$\begin{align*}
u(t) &= \varphi(t), \quad \forall t \in [-r, 0]; \\
u(T) &= B
\end{align*}$$
has been studied in [10] (see also the references therein). Another type of delay differential inclusions of the form

\[ \dot{u}(t) \in H(t, \tau(t)u), \quad \text{a.e. } t \in [0, 1] \]

with the boundary conditions

\[ u(t) = \varphi(t), \quad \forall t \in [-r, 0]; \]
\[ u(0) = u_0, \]

where, for any \( t \in [0, 1], \tau(t) : C_E([-r, t]) \to C_E([-r, 0]) \) is defined by \( \tau(t)u(s) = u(t + s) \) for all \( s \in [-r, 0] \), \( H : [0, 1] \times C_E([0, 1]) \to \mathbb{R}^n \), has been studied among others in [6, 7, 8] and [13].

In this paper, we apply the multivalued analogue of Shaefer continuous principle to prove the existence of solutions to our problem \((P_r)\). In particular, if \( F \) is uniformly Lipschitz in the sense

\[
H(F(t, x_1, y_1, z_1), F(t, x_2, y_2, z_2)) \\
\leq k_1\|x_1 - x_2\| + k_2\|y_1 - y_2\| + k_3\|z_1 - z_2\|
\]

where \( k_1, k_2, k_3 \) are positive constants satisfying \( k_1 + k_2 + k_3 < 1 \), then we show that the solution set of \((P_r)\) is a retract of \( X := C_E([-r, 1]) \cap W^{2,1}_E([0, 1]). \)

2. Existence result

In the sequel, we need the following results from [1]. See also [14] for the two point boundary value problems for second order differential equations.

**Lemma 2.1.** Let \( E \) be a separable Banach space and let \( G : [0, 1] \times [0, 1] \to \mathbb{R} \) be the function defined by the formula

(1) \[
G(t, s) = \begin{cases} 
-s & \text{if } 0 \leq s \leq t, \\
-t & \text{if } t < s \leq \theta, \\
t(s - 1)/(1 - \theta) & \text{if } \theta < s \leq 1;
\end{cases}
\]

for \( 0 \leq t < \theta \) and by

(2) \[
G(t, s) = \begin{cases} 
-s & \text{if } 0 \leq s < \theta, \\
(\theta(s - t) + s(t - 1))/(1 - \theta) & \text{if } \theta \leq s \leq t, \\
t(s - 1)/(1 - \theta) & \text{if } t < s \leq 1;
\end{cases}
\]

for \( \theta \leq t \leq 1. \)
Then the following assertions hold.

1) If \( u \in W^{2,1}_{E}([0,1]) \) with \( u(0) = 0 \) and \( u(\theta) = u(1) \), then

\[
(3) \quad u(t) = \int_{0}^{1} G(t,s) \bar{u}(s) ds, \forall t \in [0,1].
\]

2) \( G(\cdot, s) \) is derivable on \([0,1]\), for every \( s \in [0,1] \), its derivative is given by the formula

\[
(4) \quad \frac{\partial G}{\partial t}(t,s) = \begin{cases} 
0 & \text{if } 0 \leq s \leq t, \\
-1 & \text{if } t < s \leq \theta, \\
(s-1)/(1-\theta) & \text{if } \theta < s \leq 1;
\end{cases}
\]

for \( 0 \leq t < \theta \) and by

\[
(5) \quad \frac{\partial G}{\partial t}(t,s) = \begin{cases} 
0 & \text{if } 0 \leq s < \theta, \\
(s-\theta)/(1-\theta) & \text{if } \theta \leq s \leq t, \\
(s-1)/(1-\theta) & \text{if } t < s \leq 1;
\end{cases}
\]

for \( \theta \leq t \leq 1 \).

3) \( G(\cdot, \cdot) \) and \( \frac{\partial G}{\partial t}(\cdot, \cdot) \) satisfies

\[
(6) \quad \sup_{t,s \in [0,1]} |G(t,s)| \leq 1, \quad \sup_{t,s \in [0,1]} \left| \frac{\partial G}{\partial t}(t,s) \right| \leq 1.
\]

4) For \( f \in L^1_{E}([0,1]) \) and for the mapping \( u_f : [0,1] \to E \) defined by

\[
(7) \quad u_f(t) = \int_{0}^{1} G(t,s) f(s) ds, \forall t \in [0,1],
\]

one has \( u_f(0) = 0 \) and \( u_f(\theta) = u_f(1) \).

Further, the mapping \( u_f \) is derivable, and its derivative \( \dot{u}_f \) satisfies

\[
(8) \quad \lim_{h \to 0} \frac{u_f(t+h) - u_f(t)}{h} = \dot{u}_f(t) = \int_{0}^{1} \frac{\partial G}{\partial t}(t,s) f(s) ds
\]

for all \( t \in [0,1] \). Consequently, \( \dot{u}_f \) is a continuous mapping from \([0,1]\) into \( E \).
The mapping $\tilde{u}_f$ is scalarly derivable, that is, there exists a mapping $\tilde{u}_f : [0, 1] \to E$ such that, for every $x' \in E'$, the scalar function $\langle x', \tilde{u}_f(t) \rangle$ is derivable with $\frac{d}{dt} \langle x', \tilde{u}_f(t) \rangle = \langle x', \tilde{u}_f(t) \rangle$; further

$$\tilde{u}_f = f \text{ a.e. on } [0, 1].$$

\section*{Proposition 2.1}

Let $E$ be a separable Banach space and let $f : [0, 1] \to E$ be a continuous mapping (respectively a mapping in $L^1_E([0, 1])$). Then the mapping

$$u_f(t) = \int_0^1 G(t, s)f(s)ds, \forall t \in [0, 1]$$

is the unique $C^2_E([0, 1])$-solution (respectively $W^{2,1}_E([0, 1])$-solution) to the differential equation

$$\begin{cases}
\ddot{u}(t) = f(t) & \forall t \in [0, 1]; \\
u(0) = 0; \ u(1) = u(1).
\end{cases}$$

We also need the following fixed point theorem which is the multivalued analogue of the Shaefer continuation principle. For more details for the fixed point theory we refer the reader to [11].

\section*{Theorem 2.1}

Let $Y$ be a normed linear space and $A : Y \to 2^Y$ an upper semicontinuous compact multivalued operator with compact convex values. Suppose that there exists an $R > 0$ such that the a priori estimate

$$x \in \lambda Ax \quad (0 < \lambda \leq 1) \Rightarrow \|x\| \leq R$$

holds. Then $A$ has a fixed point in the ball $\overline{B}(0, R)$.

Now, we are ready to prove our main existence theorem.

\section*{Theorem 2.2}

Let $E$ be a separable Banach space, $F : [0, 1] \times E \times E \times E \to E$ be a convex compact valued multifunction, Lebesgue-measurable on $[0, 1]$ and upper semicontinuous on $E \times E \times E$. We assume that $F(t, x, y, z) \subset \Gamma(t)$ for all $(t, x, y, z) \in [0, 1] \times E \times E \times E$, for some convex norm-compact valued, and measurable multifunction $\Gamma : [0, 1] \to E$ which is integrably bounded, that is, there exists a function $k \in L^1_{\text{loc}}([0, 1])$ such that $\|v\| \leq |k(t)|$ a.e. $t \in [0, 1]$ for all $v \in \Gamma(t)$. Let $h : [0, 1] \to [-r, t]$ be a continuous mapping and $\varphi \in C_E([-r, 0])$ with $\varphi(0) = 0$. Then the boundary value problem $(\mathcal{P}_r)$ has at least one solution in $X := C_E([-r, 1]) \cap W^{2,1}_E([0, 1])$. 

Proof. We transform the problem \((\mathcal{P}_r)\) into a fixed point inclusion in the Banach space \(X\). By Lemma 2.1 and Proposition 2.2, the existence solution of \((\mathcal{P}_r)\) is equivalent to the problem of finding \(u \in X\) such that

\[
\begin{align*}
(10) & \quad \left\{ \begin{array}{l}
u(t) \in \int_0^1 G(t, s) F(t, u(s), u(h(s)), \dot{u}(s)) ds, \ \forall t \in [0, 1] \\
u(t) = \varphi(t), \ \forall t \in [-r, 0]. \end{array} \right.
\end{align*}
\]

Define the operator \(\mathcal{A}\) on \(X\) by

\[
\begin{align*}
(11) & \quad \mathcal{A}u = \{ v \in X/ \nu = \varphi \ \text{on} \ [-r, 0] \ \text{and} \\
& \quad v(t) = \int_0^1 G(t, s) f(s) ds, \ \forall t \in [0, 1], \ f \in S^1_F(u) \}
\end{align*}
\]

where

\[
(12) \quad S^1_F(u) = \{ \vartheta \in L^1_E([0, 1])/ \vartheta(t) \in F(t, u(t), u(h(t)), \dot{u}(t)), \ \text{a.e.} \ t \in [0, 1] \}.
\]

Then, the integral inclusion (10) is equivalent to the operator inclusion

\[
(13) \quad \nu(t) \in \mathcal{A}u(t), \ \forall t \in [-r, 1].
\]

It is clear that \(\mathcal{A}\) has its values in \(X\), using Lemma 2.1 and the assumption \(\varphi(0) = 0\).

Step 1. First, let us recall that the set \(S^1_E\) of all measurable selections of \(\Gamma\) is included in \(L^1_E([0, 1])\) and it is convex and compact for the weak topology \(\sigma(L^1_E([0, 1]), L^\infty_E([0, 1]))\). Furthermore, the set-valued integral

\[
\int_0^1 \Gamma(t) dt = \left\{ \int_0^1 f(t) dt, \ f \in S^1_E \right\}
\]

is convex and norm-compact. (See [4, 5, 9] for a more general result). On the other hand, let us observe that, for any Lebesgue measurable mappings \(u, w : [0, 1] \to E\), \(v : [-r, 1] \to E\), there is a Lebesgue-measurable selection \(s \in S^1_E\) such that \(s(t) \in F(t, u(t), v(h(t)), w(t))\) a.e. Indeed, there exist sequences \((u_n)\), \((v_n)\) and \((w_n)\) of simple \(E\)-valued mappings which converge pointwise to \(u\), \(v\) and \(w\) respectively, for \(E\) endowed with the
norm topology. Notice that the multifunctions \( F(., u_n(.), v_n(h()), w_n(.)) \) are Lebesgue-measurable. In view of the existence theorem of measurable selection (see [9]), for each \( n \), there is a Lebesgue-measurable selection \( s_n \) of \( F(., u_n(.), v_n(h()), w_n(.)) \). As \( s_n(t) \in F(t, u_n(t), v_n(h(t)), w_n(t)) \subset \Gamma(t) \), for all \( t \in [0, 1] \) and as \( S^1_F \) is weakly compact in \( L^1_E([0, 1]) \), by Eberlein-Šmulian theorem, we may extract from \( (s_n) \) a subsequence \( (s'_n) \) which converges \( \sigma(L^1_E([0, 1]), L^\infty_E([0, 1])) \) to a mapping \( s \in S^1_F \). An application of the Banach-Mazur's trick to \( (s'_n) \) provides a sequence \( (z_n) \) with \( z_n \in co\{s_{k} : k \geq n \} \) such that \( (z_n) \) converges pointwise almost everywhere to \( s \). Using this fact and the pointwise convergence of the sequences \( (u_n), (v_n) \) and \( (w_n) \) and the upper semicontinuity of \( F(t, ., ., .) \) it is not difficult to see that \( s(t) \in F(t, u(t), v(h(t)), w(t)) \) a.e. Consequently, \( S^1_F(u) \neq \emptyset \) for all \( u \in X \). This shows that \( A \) is well defined.

**Step 2.** In this step we will show that the multivalued operator \( A \) satisfies all the conditions of Theorem 2.1. Clearly, \( Au \) is convex for each \( u \in X \). First, we show that \( A \) has compact values on \( X \). For each \( u \in X \), let \( (v_n) \) be a sequence in \( Au \), then by (11), for every \( n \) there exists \( f_n \in S^1_F(u) \subset S^1_F \) such that

\[
v_n(t) = \int_0^1 G(t, s)f_n(s)ds, \quad \forall t \in [0, 1]
\]

and \( v_n(t) = \varphi(t) \) for all \( t \in [-r, 0] \). Since \( S^1_F \) is weakly compact in \( L^1_E([0, 1]) \), we may extract from \( (f_n) \) a subsequence (that we do not relabel) converging \( \sigma(L^1_E, L^\infty_E) \) to a mapping \( f \in S^1_F \). Since \( F(t, ., ., .) \) is upper semicontinuous and has convex compact values, we get \( f(t) \in F(t, u(t), u(h(t)), u(t)) \) for almost every \( t \in [0, 1] \). In particular, for every \( x' \in E^* \) and for every \( t \in [0, 1] \), we have

\[
\lim_{n \to \infty} \langle x', \int_0^1 G(t, s)f_n(s)ds \rangle = \lim_{n \to \infty} \int_0^1 \langle G(t, s)x', f_n(s) \rangle ds
\]

\[
= \int_0^1 \langle G(t, s)x', f(s) \rangle ds
\]

\[
= \langle x', \int_0^1 G(t, s)f(s)ds \rangle.
\]

(14)

As the set-valued integral \( \int_0^1 G(t, s)\Gamma(s)ds \) \( (t \in [0, 1]) \) is norm compact, (14) shows that the sequence \( (v_n(.)) = (\int_0^1 G(\cdot, s)f_n(s)ds) \) converges pointwise
to \( v(.) = \int_0^1 G(.,s) f(s) ds \), for \( E \) endowed with the strong topology. At this point, it is worth mentioning that the sequence \( (\hat{v}_n)(.) = (\int_0^1 \frac{\partial G}{\partial t}(.,s) f_n(s) ds) \) converges pointwise to \( \hat{v}(.) \), for \( E \) endowed with the strong topology, using as above the weak convergence of \( (f_n) \) and the norm compactness of the set-valued integral \( \int_0^1 \frac{\partial G}{\partial t}(t,s) \Gamma(s) ds \). Hence \( (v_n) \) converges in \( X \) to a mapping \( w \) where

\[
w(t) = \int_0^1 G(t,s) f(s) ds, \quad \forall t \in [0,1]
\]

and \( w(t) = \varphi(t) \) for all \( t \in [-r,0] \). This says that \( \mathcal{A}u \) is compact in \( X \).

Next, we show that \( \mathcal{A} \) is a compact operator, that is, \( \mathcal{A} \) maps bounded sets into relatively compact sets in \( X \). Let \( S \) be a bounded set in \( X \) and let \( u \in S \), for each \( v \in \mathcal{A}u \) there exists \( f \in S_k^1(u) \) such that

\[
v(t) = \int_0^1 G(t,s) f(s) ds, \quad \forall t \in [0,1]
\]

and \( v(t) = \varphi(t) \) for all \( t \in [-r,0] \). Observe that for all \( t, t' \in [0,1] \)

\[
\|v(t) - v(t')\| \leq \int_0^1 |G(t,s) - G(t',s)| \|f(s)\| ds
\]

\[
\leq \int_0^1 |G(t,s) - G(t',s)| |k(s)| ds,
\]

and

\[
\|\dot{v}(t) - \dot{v}(t')\| \leq \int_0^1 \left| \frac{\partial G}{\partial t}(t,s) - \frac{\partial G}{\partial t}(t',s) \right| |k(s)| ds.
\]

The function \( G \) is continuous on the compact set \( [0,1] \times [0,1] \), so it is uniformly continuous there. In addition, \( k \in L_k^1([0,1]) \), then, the right-hand side of the above inequalities tends to 0 as \( t \to t' \). We conclude that \( \mathcal{A}(S) \) and \( \{\dot{v} : v \in \mathcal{A}(S)\} \) are equicontinuous in \( C_E([0,1]) \). Since \( \varphi \in C_E([-r,0]) \) we get the equicontinuity of \( \mathcal{A}(S) \) in \( X \). Further, for each \( t \in [-r,1] \) and each \( \tau \in [0,1] \), the sets \( \mathcal{A}(S)(t) = \{v(t) : v \in \mathcal{A}(S)\} \) and \( \{\dot{v}(\tau) : v \in \mathcal{A}(S)\} \) are relatively compact in \( E \) because they are included in the norm compact sets \( \int_0^1 G(t,s) \Gamma(s) ds \) and \( \int_0^1 \frac{\partial G}{\partial t}(t,s) \Gamma(s) ds \), respectively. An application of the Arzelà-Ascoli theorem implies that \( \mathcal{A}(S) \) is relatively compact in \( X \) and hence \( \mathcal{A} \) is compact.
Next, we prove that the graph of $\mathcal{A}$, $\text{gph}(\mathcal{A}) = \{(u, v) \in \mathbf{X} \times \mathbf{X} / v \in \mathcal{A}u\}$ is closed. Let $(u_n, v_n)$ be a sequence of $\text{gph}(\mathcal{A})$ converging uniformly to $(u, v) \in \mathbf{X} \times \mathbf{X}$ with respect to $\| \cdot \|_{\mathbf{X}}$. Since $v_n \in \mathcal{A}u_n$, for each $n$ there exists $f_n \in S_f^1(u_n) \subset S_f^1$ such that

$$v_n(t) = \int_0^1 G(t, s)f_n(s)ds, \quad \forall t \in [0, 1]$$

and $v_n(t) = \varphi(t)$ for all $t \in [-r, 0]$. As $S_f^1$ is weakly compact in $L^1_k([0, 1])$, we may extract from $(f_n)$ a subsequence (that we do not relabel) converging $\sigma(L^1_k, L^\infty)$ to a mapping $f \in S_f^1$.

Observe that $f_n(t) \in F(t, u_n(t), u_n(h(t)), \dot{u}_n(t))$. Since $\|u_n - u\|_{\mathbf{X}} \to 0$ and $F(t, \cdot, \cdot, \cdot)$ is upper semicontinuous on $E \times E \times E$ with convex compact values we conclude that $f(t) \in F(t, u(t), u(h(t)), \dot{u}(t))$, using a closure type theorem (see [9]). Equivalently, $f \in S_f^1(u)$. On the other hand, repeating the arguments given above, it is not difficult to see that the sequence $(v_n(\cdot)) = (\int_0^1 G(\cdot, s)f_n(s)ds)$ converges pointwise to $\int_0^1 G(\cdot, s)f(s)ds$ and that the sequence $(\dot{v}_n(\cdot)) = (\int_0^1 \frac{\partial G}{\partial t}(\cdot, s)f_n(s)ds)$ converges pointwise to $\int_0^1 \frac{\partial G}{\partial t}(\cdot, s)f(s)ds$, for $E$ endowed with the strong topology. As $(v_n)$ converges to $v$ in $(\mathbf{X}, \| \cdot \|_{\mathbf{X}})$ we get

$$v(t) = \int_0^1 G(t, s)f(s)ds, \quad \forall t \in [0, 1]$$

and $v(t) = \varphi(t)$ for all $t \in [-r, 0]$. This shows that $\mathcal{A}$ has a closed graph and hence it is an upper semicontinuous operator on $\mathbf{X}$. Finally, we show that there exists an $R > 0$ such that the a priori estimate

$$u \in \lambda \mathcal{A}u \quad (0 < \lambda \leq 1) \Rightarrow \|u\| \leq R$$

holds. We have

$$u \in \lambda \mathcal{A}u \quad \Leftrightarrow \quad \text{there exists } f \in S_f^1(u) \subset S_f^1,$$

such that

$$\begin{cases}
    u(t) = \lambda \int_0^1 G(t, s)f(s)ds, \quad \forall t \in [0, 1] \\
    u(t) = \lambda \varphi(t), \quad \forall t \in [-r, 0].
\end{cases}$$
For each $t \in [0, 1]$, using relation (6) and the assumption over $\Gamma$, we have

$$
\|u(t)\| \leq \int_0^1 \|G(t, s)\| f(s)\|ds,
$$

$$
\leq \int_0^1 |k(s)|ds = \|k\|_{L^1([0, 1])}
$$
and

$$
\|\dot{u}(t)\| \leq \int_0^1 \left| \frac{\partial G}{\partial t}(t, s) \right| f(s)\|ds \leq \|k\|_{L^1([0, 1])}.
$$

On the other hand, for each $t \in [-r, 0]$ we have

$$
\|u(t)\| = \|\lambda \varphi(t)\| \leq \|\varphi\|_{C_E([-r, 0])}.
$$

Taking the above inequalities into account, we obtain

$$
\|u\|_X \leq \max \left( \|k\|_{L^1([0, 1])}, \|\varphi\|_{C_E([-r, 0])} \right) = R.
$$

Hence by the conclusion of Theorem 2.1, $A$ has a fixed point in the ball $B(0, R)$, what, in turn, means that this point is a solution in $X$ to our boundary value problem $(P_r)$.

To end the paper, we prove below that under suitable Lipschitz assumption on the second member, the solution set of $(P_r)$ is a retract of $X$. Compare with Theorem 1 in [2], and Theorem 5 in [12] in which the authors deal with nonconvex differential inclusions and Theorem 2 in [2] in the convex case. See also [3].

**Theorem 2.3.** Under the hypotheses of Theorem 2.2, if we replace the upper semicontinuity assumption on $F(t, \cdot, \cdot, \cdot)$ by the condition

$$
(\ast) \quad \mathcal{H}(F(t, x_1, y_1, z_1), F(t, x_2, y_2, z_2))
\leq k_1 \|x_1 - x_2\| + k_2 \|y_1 - y_2\| + k_3 \|z_1 - z_2\|
$$

for all $(t, x_1, y_1, z_1), (t, x_2, y_2, z_2) \in [0, 1] \times E \times E \times E$, where $k_1, k_2, k_3$ are positive constants satisfying $k_1 + k_2 + k_3 < 1$. Then the solution set of the problem $(P_r)$ is a retract of $X$. 
Proof. The idea of proof comes from ([2], Theorem 2). Let us denote by \( \mathcal{X}(\varphi) \) the solution set of \( (P_r) \). By virtue of the proof of Theorem 2.2, \( u \in \mathcal{X}(\varphi) \) iff \( u \in \mathcal{A}u \). Let us prove that \( \mathcal{A} \) is a contraction. Let \( u_1, u_2 \in \mathbf{X} \) and \( v_1 \in \mathcal{A}u_1 \), then \( v_1 = \varphi \) on \([-r, 0]\) and there exists \( f_1 \in S_F(u_1) \) such that \( v_1(t) = \int_0^t G(t, s)f_1(s)ds \), for all \( t \in [0, 1] \). We have \( f_1(t) \in F(t, u_1(t), u_1(h(t)), \dot{u}_1(t)) \), as \( F \) is compact valued and \( F(\cdot, u_2(\cdot), u_2(h(\cdot)), \dot{u}_2(\cdot)) \) is measurable, the multifunction \( H \) defined from \([0, 1]\) into \( E \) by

\[
H(t) = \{ w \in F(t, u_2(t), u_2(h(t)), \dot{u}_2(t)) : \|f_1(t) - w\| = d(f_1(t), F(t, u_2(t), u_2(h(t)), \dot{u}_2(t))) \text{ a.e} \}
\]

is also measurable with nonempty closed values. In view of the existence theorem of measurable selections (See [9]), there is a measurable mapping \( f_2 : [0, 1] \to E \) such that \( f_2(t) \in H(t) \) for all \( t \in [0, 1] \). This yields \( f_2(t) \in F(t, u_2(t), u_2(h(t)), \dot{u}_2(t)) \) and

\[
(15) \quad \|f_1(t) - f_2(t)\| = d(f_1(t), F(t, u_2(t), u_2(h(t)), \dot{u}_2(t))) \text{ a.e on } [0, 1].
\]

Let us define the mapping \( v_2 \) on \([-r, 1]\) by

\[
v_2(t) = \begin{cases} \varphi(t), & \forall t \in [-r, 0] \\ \int_0^1 G(t, s)f_2(s)ds, & \forall t \in [0, 1]. \end{cases}
\]

Clearly, \( v_2 \in \mathcal{A}u_2 \). For every \( t \in [0, 1] \) we have

\[
\|v_1(t) - v_2(t)\| = \left\| \int_0^1 G(t, s)(f_1(s) - f_2(s))ds \right\| \leq \int_0^1 \|f_1(s) - f_2(s)\|ds.
\]

From this, (15) and the assumption \((*)\), for every \( t \in [0, 1] \) we obtain

\[
\|v_1(t) - v_2(t)\|
\leq \int_0^1 \|f_1(s) - f_2(s)\|ds
= \int_0^1 d(f_1(s), F(s, u_2(s), u_2(h(s)), \dot{u}_2(s)))ds
\]
\begin{align*}
\leq & \int_0^1 \mathcal{H}(F(s, u_1(s), u_1(h(s)), \dot{u}_1(s)), F(s, u_2(s), u_2(h(s)), \dot{u}_2(s))) ds \\
\leq & \int_0^1 (k_1 \|u_1(s) - u_2(s)\| + k_2 \|u_1(h(s)) - u_2(h(s))\| + k_3 \|\dot{u}_1(s) - \dot{u}_2(s)\|) ds \\
\leq & \int_0^1 (k_1 \|u_1 - u_2\| + k_2 \|u_1 - u_2\| + k_3 \|u_1 - u_2\|) ds \\
= & (k_1 + k_2 + k_3) \|u_1 - u_2\| \|x\|.
\end{align*}

Consequently,

\begin{equation}
(16) \quad \|v_1 - v_2\|_{C_{\mathcal{E}}([-r, 1])} = \|v_1 - v_2\|_{C_{\mathcal{E}}([0,1])} \leq (k_1 + k_2 + k_3) \|u_1 - u_2\| \|x\|.
\end{equation}

On the other hand, using Lemma 2.1, we have

\begin{align*}
\|\dot{v}_1(t) - \dot{v}_2(t)\| = & \left\| \int_0^1 \frac{\partial G}{\partial t}(t, s)(f_1(s) - f_2(s)) ds \right\| \\
\leq & \int_0^1 \|f_1(s) - f_2(s)\| ds.
\end{align*}

By repeating the same arguments, we obtain

\begin{equation}
(17) \quad \|\dot{v}_1 - \dot{v}_2\|_{C_{\mathcal{E}}([0,1])} \leq (k_1 + k_2 + k_3) \|u_1 - u_2\| \|x\|.
\end{equation}

The inequalities (16) and (17) give

\begin{align*}
\|v_1 - v_2\| \|x\| \leq (k_1 + k_2 + k_3) \|u_1 - u_2\| \|x\|.
\end{align*}

Then we get

\begin{align*}
d(v_1, Au_2) = \inf_{v_2 \in Au_2} \|v_1 - v_2\| \|x\| \leq (k_1 + k_2 + k_3) \|u_1 - u_2\| \|x\|,
\end{align*}

and

\begin{align*}
\sup_{v_1 \in Au_2} d(v_1, Au_2) \leq (k_1 + k_2 + k_3) \|u_1 - u_2\| \|x\|.
\end{align*}
By similar computations and by interchanging the role of $u_1$ and $u_2$, we have

$$\sup_{v_2 \in Au_2} d(v_2, Au_1) \leq (k_1 + k_2 + k_3)\|u_1 - u_2\|_X.$$ 

Hence we obtain

$$\mathcal{H}(Au_1, Au_2) \leq (k_1 + k_2 + k_3)\|u_1 - u_2\|_X$$

with $(k_1 + k_2 + k_3) < 1$, proving that $A$ is a contraction in $X$. By a result of Ricceri [19], the set

$$\text{Fix}(A) = \{ u \in X : u \in Au \}$$

is a retract of $X$. It is clear that $\text{Fix}(A) = \mathcal{X}(\varphi)$. This shows that the solution set of $(Pr)$ is a retract of $X$ and the proof of the theorem is complete.

References


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