STRONG $f$-STAR FACTORS OF GRAPHS

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Abstract
Let $G$ be a graph and $f : V(G) \rightarrow \{2, 3, \ldots\}$. A spanning subgraph $F$ is called strong $f$-star of $G$ if each component of $F$ is a star whose center $x$ satisfies $\deg_F(x) \leq f(x)$ and $F$ is an induced subgraph of $G$. In this paper, we prove that $G$ has a strong $f$-star factor if and only if $\text{oddca}(G - S) \leq \sum_{x \in S} f(x)$ for all $S \subset V(G)$, where $\text{oddca}(G)$ denotes the number of odd complete-cacti of $G$.

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1. Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph $G$, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. We write $|G|$ for the order of $G$ (i.e., $|G| = |V(G)|$). For a vertex $v$ of $G$, we denote by $\deg_G(v)$ the degree of $v$ in $G$. For a vertex set $S$ of $G$, let $G - S$ denote the subgraph of $G$ induced by $V(G) - S$. Let $\text{Iso}(G)$ and $\text{iso}(G)$ denote the set of isolated vertices and the number of isolated vertices of $G$, respectively.

A graph $G$ is called a complete-cactus if $G$ is connected and every block of $G$ is a complete graph. A complete-cactus is called an odd complete-cactus if all its blocks are complete graphs of odd order. Note that $K_1$ is an odd complete-cactus.

For a set $\mathcal{S}$ of connected graphs, a spanning subgraph $F$ of a graph $G$ is called an $\mathcal{S}$-factor of $G$ if each component of $F$ is isomorphic to an element of $\mathcal{S}$. A complete bipartite graph $K_{1,n}$ is called a star, and its vertex of degree $n$ is called the center. For $K_{1,1}$, an arbitrarily chosen vertex is its center.

The following theorem was independently obtained by Las Vergnas [6] and by Amahashi and Kano [2].
Theorem 1 [2, 6]. Let $n \geq 2$ be an integer. Then a graph $G$ has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor if and only if $\text{iso}(G - S) \leq n|S|$ for all $S \subset V(G)$.

Let $G$ be a graph and let $f : V(G) \to \{2, 3, 4, \ldots\}$ be a function defined on $V(G)$. Then a spanning subgraph $F$ is called an $f$-star factor of $G$ if each component of $F$ is a star and its center $x$ satisfies $\text{deg}_F(x) \leq f(x)$. The following theorem gives a criterion for a graph to have an $f$-star factor.

Theorem 2 [3]. Let $G$ be a graph and let $f : V(G) \to \{2, 3, \ldots\}$ be a function. Then $G$ has an $f$-star factor if and only if $\text{iso}(G - S) \leq \sum_{x \in S} f(x)$ for all $S \subset V(G)$.

For a set $S$ of connected graphs, a subgraph $H$ of $G$ is called a strong $S$-subgraph if every component of $H$ is isomorphic to an element of $S$ and is an induced subgraph of $G$. A spanning strong $S$-subgraph is called a strong $S$-factor. A strong $\{K_{1,1}, K_{1,2}, \ldots\}$-factor is briefly called a strong star factor. Kelmans [7] and Saito and Watanabe [8] proved independently the following theorem.

Theorem 3 [7, 8]. A connected graph $G$ has a strong star factor if and only if $G$ is not an odd complete-cactus.

For a graph $G$, let $\text{OddCa}(G)$ denote the set of components of $G$ that are odd complete-cacti, and let $\text{oddca}(G) = |\text{OddCa}(G)|$ denote the number of odd complete-cacti of $G$. Egawa, Kano and Kelmans [4] generalized the above theorem as follows by considering a strong $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor.

Theorem 4 [4]. Let $n \geq 2$ be an integer. Then a graph $G$ has a strong $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor if and only if $\text{oddca}(G - S) \leq n|S|$ for all $S \subset V(G)$.

A subgraph $H$ is called a strong $f$-star subgraph of $G$ if each component of $H$ is a star, whose center $x$ satisfies $\text{deg}_H(x) \leq f(x)$, and $H$ is an induced subgraph of $G$. A spanning $f$-star subgraph of $G$ is called a strong $f$-star factor of $G$. Obviously, if $f(x) = n$ for all $x \in V(G)$, then a strong $f$-star factor of $G$ is a strong $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor. In this paper, we obtain the following result which is a generalization of Theorem 4.

Theorem 5. Let $G$ be a graph and let $f : V(G) \to \{2, 3, \ldots\}$ be a function. Then $G$ has a strong $f$-star factor if and only if

\[
\text{oddca}(G - S) \leq \sum_{x \in S} f(x), \text{ for all } S \subset V(G).
\]

A strong $f$-star subgraph $H$ of a graph $G$ is said to be maximum if $G$ has no strong $f$-star subgraph $H'$ such that $|H'| > |H|$. A formula for the order of a maximum strong $f$-star subgraph of a graph is easily obtained as a maximum matching, which is given in the following theorem.
Theorem 6. Let \( G \) be a graph and let \( f : V(G) \to \{2, 3, 4, \ldots\} \) be a function. Then the order of a maximum strong \( f \)-star subgraph \( H \) of \( G \) is given by
\[
|H| = |G| - \max_{X \subset V(G)} \left\{ \text{oddca}(G - X) - \sum_{x \in X} f(x) \right\}.
\]

Finally, we consider a problem of covering a given vertex subset with a strong \( f \)-star subgraph. The condition for the existence of such a subgraph, which is given in the following theorem, is a natural extension of the criterion for the existence of a strong \( f \)-star factor.

Theorem 7. Let \( G \) be a graph and let \( f : V(G) \to \{2, 3, 4, \ldots\} \) be a function. Let \( W \) be a subset of \( V(G) \). Then \( G \) has a strong \( f \)-star subgraph covering \( W \) if and only if
\[
\text{oddca}(G - S|W) \leq \sum_{x \in S} f(x), \text{ for all } S \subset V(G),
\]
where \( \text{oddca}(G - S|W) \) denotes the number of odd complete-cacti of \( G - S \) contained in \( W \).

2. Proof of the Results

We need some other notations. For two sets \( X \) and \( Y \), \( X \subset Y \) means that \( X \) is a proper subset of \( Y \). Let \( G \) be a graph. For two vertices \( x \) and \( y \) of \( G \), we write \( xy \) or \( yx \) for an edge joining \( x \) to \( y \). For a vertex \( v \) of \( G \), we denote by \( N_G(v) \) the neighborhood of \( v \). For a subset \( S \) of \( V(G) \), we define \( N_G(S) := \bigcup_{x \in S} N_G(x) \). For convenience, we briefly call a complete-cactus a cactus in the following proofs. Analogously, an odd complete-cactus is called an odd cactus. Every block of a cactus is a complete graph, and we call it an odd block or even block according to its order.

In order to prove Theorem 5, we need the following lemmas.

Lemma 8 [4]. (i) Let \( G \) be an odd complete-cactus. Then for every vertex \( v \) of \( G \), \( G - v \) has a 1-factor.

(ii) An odd complete-cactus does not have a strong star factor.

Lemma 9 [5]. Let \( G \) be a bipartite graph with bipartition \((A, B)\), and let \( g, f : V(G) \to \mathbb{Z} \) be functions such that \( g(x) \leq f(x) \) for all \( x \in V(G) \). Then \( G \) has a \((g, f)\)-factor if and only if
\[
\gamma^*(X,Y) = \sum_{x \in X} f(x) + \sum_{x \in Y} (\deg_G(x) - g(x)) - e_G(X, Y) \geq 0,
\]
and
\[
\gamma^*(Y,X) = \sum_{x \in X} f(x) + \sum_{x \in Y} (\deg_G(x) - g(x)) - e_G(Y, X) \geq 0,
\]
for all subsets \( X \subseteq A \) and \( Y \subseteq B \).
Lemma 10. Let $G$ be a bipartite graph with bipartition $(A, B)$. Let $f : V(G) \to \{1, 2, 3, \ldots \}$ be a function such that $f(x) \geq 2$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$. Then $G$ has a $(1, f)$-factor if and only if

\begin{equation}
|N_G(X)| \geq |X| \text{ for all } X \subseteq A, \text{ and } \sum_{x \in N_G(Y)} f(x) \geq |Y| \text{ for all } Y \subseteq B.
\end{equation}

Proof. If $G$ has a $(1, f)$-factor $F$, then (4) follows from

\begin{equation}
|N_G(X)| \geq |N_F(X)| \geq |X| \text{ and } \sum_{x \in N_G(Y)} f(x) \geq \sum_{x \in N_F(Y)} f(x) \geq |Y|.
\end{equation}

Conversely, assume that (4) holds. We may assume that $G$ is connected, since otherwise each component satisfies (4) and has a $(1, f)$-factor by induction, and hence $G$ itself has a $(1, f)$-factor. For any subsets $X \subseteq A$ and $Y \subseteq B$, it follows from (4) that

\begin{equation}
\gamma^+(X, Y) = \sum_{x \in X} f(x) + \sum_{y \in Y} (\deg_G - x(y) - 1) \\
\geq \sum_{x \in X} f(x) - |S| \geq \sum_{x \in N_G(S)} f(x) - |S| \geq 0,
\end{equation}

where $S = \text{Iso}(G - X) \cap Y \subseteq B$, $N_G(S) \subseteq X$.

\begin{equation}
\gamma^*(Y, X) = \sum_{y \in Y} f(y) + \sum_{x \in X} (\deg_G - y(x) - 1) \\
\geq |Y| - |T| \geq |N_G(T)| - |T| \geq 0,
\end{equation}

where $T = \text{Iso}(G - Y) \cap X \subseteq A$, $N_G(T) \subseteq Y$.

Therefore by Lemma 9, $G$ has the desired $(1, f)$-factor. \hfill \blacksquare

Proof of Theorem 5. Suppose that $G$ has a strong $f$-star factor $F$. Let $\emptyset \neq S \subset V(G)$. Since every odd cactus $D$ of $G - S$ does not have a strong $f$-star by Lemma 8, $F$ has an edge joining $D$ to $S$. It is obvious that for every vertex $s \in S$, $F$ has at most $f(x)$ edges joining $x$ to odd cacti in $G - S$. Hence $\text{oddca}(G - S) \leq \sum_{x \in S} f(x)$.

We shall prove the sufficiency of Theorem 5 by induction on $\sum_{x \in V(G)} f(x)$. We may assume that $|G| \geq 3$ and $G$ is connected, since otherwise by applying the induction hypothesis to each component, we can obtain the desired strong star-factor of $G$. By taking $S = \emptyset$, it follows that $G$ is not an odd cactus.

Obviously, $\sum_{x \in V(G)} f(x) \geq 2|G|$, since $f(x) \geq 2$ for all $x \in V(G)$. If $\sum_{x \in V(G)} f(x) = 2|G|$, then $f(x) = 2$ for all $x \in V(G)$. Thus the condition (1) becomes $\text{oddca}(G - S) \leq 2|S|$, for all $S \subset V(G)$. 

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By Theorem 4, $G$ has a strong $\{K_{1,1}, K_{1,2}\}$-factor, which is the desired strong $f$-factor of $G$. So we may assume that $\sum_{x \in V(G)} f(x) \geq 2|G| + 1$. Then there exists a vertex $w \in V(G)$ such that $f(w) \geq 3$.

Let us define the number $\beta$ by

$$\beta = \min_{\emptyset \neq X \subset V(G)} \left\{ \sum_{x \in X} f(x) - \text{oddca}(G - X) \right\}.$$ 

Then $\beta \geq 0$ by (1), and it follows from the definition of $\beta$ that

(5) \hspace{1cm} \text{oddca}(G - Y) \leq \sum_{x \in Y} f(x) - \beta, \text{ for all } \emptyset \neq Y \subset V(G).$

Take a maximal subset $S$ of $V(G)$ such that

(6) \hspace{1cm} \sum_{x \in S} f(x) - \text{oddca}(G - S) = \beta.

Claim 1. $\beta = 0$.

Proof. Suppose that $\beta \geq 1$. Define $f^* : V(G) \to \{2, 3, 4,\ldots\}$ by

$$f^*(x) = \begin{cases} f(x) - 1 & \text{if } x = w, \\ f(x) & \text{otherwise.} \end{cases}$$

Let $\emptyset \neq X \subset V(G)$. Then we have

$$\text{oddca}(G - X) \leq \sum_{x \in X} f(x) - \beta \leq \sum_{x \in X} f(x) - 1 \leq \sum_{x \in X} f^*(x).$$

Hence, $G$ has a strong star-factor $F^*$ with respect to $f^*$ by induction, which is also the strong $f$-star factor of $G$. \qed

Hereafter we assume $\beta = 0$.

Claim 2. Every component of $G - S$ which is not an odd cactus has a strong $f$-star factor.

Proof. Let $D$ be a component of $G - S$ which is not an odd cactus, and let $\emptyset \neq X \subset V(D)$. Then by (5), we have

$$\text{oddca}(G - S) + \text{oddca}(D - X) = \text{oddca}(G - S \cup X)$$

$$\leq \sum_{x \in S \cup X} f(x) = \sum_{x \in S} f(x) + \sum_{x \in X} f(x).$$

Thus $\text{oddca}(D - X) \leq \sum_{x \in X} f(x)$ by (6), which implies that $D$ has a strong $f$-star factor by induction. \qed
We construct a bipartite graph $B$ with bipartition $(S, \text{OddCa}(G - S))$ in which two vertices $x \in S$ and a component $C \in \text{OddCa}(G - S)$ are joined by an edge of $B$ if and only if $x$ is adjacent to $C$ in $G$.

**Claim 3.** For every $\emptyset \neq X \subseteq S$ and $\emptyset \neq Y \subseteq \text{OddCa}(G - S)$, it follows that $|N_B(X)| \geq |X|$ and $\sum_{x \in N_B(Y)} f(x) \geq |Y|$.

**Proof.** Let $\emptyset \neq X \subseteq S$. By (5) and $\beta = 0$, we obtain

$$
\sum_{x \in S - X} f(x) \geq \text{oddca}(G - (S - X)) \geq \text{oddca}(G - S) - |N_B(X)|
$$

$$
\geq \sum_{x \in S} f(x) - |N_B(X)|,
$$

which means $|N_B(X)| \geq \sum_{x \in X} f(x) \geq |X|$. Let $\emptyset \neq Y \subseteq \text{OddCa}(G - S)$. Then $N_B(Y) \subseteq S$, and by (1) we have

$$
|Y| \leq \text{oddca}(G - N_B(Y)) \leq \sum_{x \in N_B(Y)} f(x).
$$

Therefore Claim 3 holds.

By Claim 3, $B$ has a strong $f$-star factor $H$ given in Lemma 10, which is a $(1, f)$-factor with minimal edge set, and every component of $\text{OddCa}(G - S)$ has degree one in $H$. Consequently, by Lemma 8(i) and Claim 2, we can obtain a strong $f$-star factor of $G$ from $H$. 

**Proof of Theorem 6.** Let $d = \max_{x \in V(G)} \{\text{oddca}(G - X) - \sum_{x \in X} f(x)\}$. Then $d \geq 0$ by considering the case $X = \emptyset$. Moreover, if $d = 0$, then (2) follows from Theorem 5. Hence we may assume $d \geq 1$. Let $S$ be a subset of $V(G)$ such that

$$
\text{oddca}(G - S) - \sum_{x \in S} f(x) = d.
$$

Then by considering $(S \cup \text{OddCa}(G - S))_G$, which is the subgraph of $G$ induced by $S \cup \text{OddCa}(G - S)$, we have that every strong $f$-star subgraph of $G$ cannot cover at least $\text{oddca}(G - S) - \sum_{x \in S} f(x)$ odd cacti of $\text{OddCa}(G - S)$. Hence $|H| \leq |G| - d$, when $H$ is a maximum strong $f$-star subgraph of $G$.

Next we prove the inverse inequality $|H| \geq |G| - d$ for a maximum strong $f$-star subgraph $H$ of $G$. Add $2d$ new vertices $\{v_i, u_i : 1 \leq i \leq d\}$ together with $d$ new edges $\{v_iu_i : 1 \leq i \leq d\}$ to $G$. Then join every $v_i$ to every vertex of $G$ by new edges. Denote the resulting graph by $G^*$, and define a function $f^* : V(G^*) \to \{2, 3, 4, \ldots\}$ by $f^*(v_i) = f^*(u_i) = 2$ for all $1 \leq i \leq d$, and $f^*(x) = f(x)$ for all $x \in V(G)$.

Let $Y$ be a non-empty subset of $V(G^*)$. We may assume that $Y$ contains no vertices of $\{u_1, \ldots, u_d\}$, when we estimate $\text{oddca}(G^* - Y)$. If $|\{v_1, \ldots, v_d\} \cap Y| < d$, then

$$
\text{oddca}(G^* - Y) \leq |Y \cap \{v_1, \ldots, v_d\}| + 1 \leq \sum_{x \in Y} f(x).
$$
If \( \{v_1, \ldots, v_d\} \subset Y \), then all the vertices of \( \{u_1, \ldots, u_d\} \) become isolated vertices of \( G^* - Y \), and so by the definition of \( d \), we obtain

\[
\text{oddca}(G^* - Y) \leq \text{oddca}(G - (Y \cap V(G))) + d
\]

\[
\leq \sum_{x \in Y \cap V(G)} f(x) + d = \sum_{x \in Y} f(x).
\]

Hence by Theorem 5, \( G^* \) has a strong \( f \)-star factor \( F^* \). Then \( H = F^* - \{u_i, v_i : 1 \leq i \leq d\} \) is a strong \( f \)-star subgraph of \( G \), which covers at least \( |G| - d \) vertices. Hence \( |H| \geq |G| - d \). Consequently, the theorem is proved.

**Proof of Theorem 7.** First suppose that \( G \) has a strong \( f \)-star subgraph \( F \) covering \( W \). Then for every odd cactus \( C \) of \( G - S \) contained in \( W \), \( F \) has at least one edge joining \( C \) to \( S \). Hence \( \text{oddca}(G - S|W) \leq \sum_{x \in S} \text{deg}_F(x) \).

Next we assume that (3) holds. We may assume that \( G \) is connected, since otherwise, by applying the induction hypothesis to each component of \( G \), we can obtain the desired factor of \( G \). By Theorem 5, we may assume that \( W \) is a proper subset of \( V(G) \), and so \( V(G) - W \neq \emptyset \). Let \( n = |V(G) - W| \). We construct a new graph \( H \) from \( G \) by adding two new vertices \( w_1, w_2 \) and by joining \( w_i (i = 1, 2) \) to every vertex in \( V(G) - W \). Define \( f^* : V(H) \to \{2, 3, \ldots\} \) by

\[
f^*(x) = \begin{cases} f(x) & \text{if } x \in V(G), \\ \max\{2, n\} & \text{if } x \in \{w_1, w_2\}. \end{cases}
\]

It is easy to see that \( G \) has a strong \( f \)-star subgraph covering \( W \) if and only if \( H \) has a strong \( f^* \)-star factor.

Let \( X \subset V(H) \). If \( w_1, w_2 \in X \), let \( S = X - \{w_1, w_2\} \), then

\[
\text{oddca}(H - X) \leq \text{oddca}(G - S|W) + n \leq \sum_{x \in S} f(x) + n < \sum_{x \in X} f^*(x).
\]

If \( w_1 \in X \) and \( w_2 \notin X \), let \( S = X - \{w_1\} \), then

\[
\text{oddca}(H - X) \leq \text{oddca}(G - S|W) + 1 \leq \sum_{x \in S} f(x) + 1 < \sum_{x \in X} f^*(x).
\]

If \( w_1, w_2 \notin X \), then

\[
\text{oddca}(H - X) = \text{oddca}(G - X|W) \leq \sum_{x \in X} f(x).
\]

Therefore, by Theorem 5, \( H \) has a strong \( f^* \)-star factor, and thus \( G \) has the desired strong \( f \)-star subgraph which covers \( W \).
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