LATTICES OF RELATIVE COLOUR-FAMILIES
AND ANTIVARIETIES

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Abstract

We consider general properties of lattices of relative colour-families and antivarieties. Several results generalise the corresponding assertions about colour-families of undirected loopless graphs, see [1]. Conditions are indicated under which relative colour-families form a lattice. We prove that such a lattice is distributive. In the class of lattices of antivarieties of relation structures of finite signature, we distinguish the most complicated (universal) objects. Meet decompositions in lattices of colour-families are considered. A criterion is found for existence of irredundant meet decompositions. A connection is found between meet decompositions and bases for anti-identities.

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1. Preliminary facts

Throughout the article, by a structure we mean a relation structure of a fixed signature $\sigma = (r_j)_{j \in J}$. A structure is said to be finite if its universe is a finite set. A homomorphism from a structure $A$ into a structure $B$ is a map $\varphi : A \to B$ such that $(\varphi(a_1), \ldots, \varphi(a_n)) \in r^B_j$ for all $j \in J$ and

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$a_1, \ldots, a_n \in A$ with $(a_1, \ldots, a_n) \in r^A_j$. If there exists a homomorphism from $A$ into $B$ then we write $A \to B$; otherwise, we write $A \not\to B$.

For every class $K$, let $K_f$ denote the set of isomorphism types of finite structures in $K$. On $K_f$, define an equivalence relation $\equiv$ as follows: $A \equiv B$ if and only if $A \to B$ and $B \to A$. The relation $\to$ induces a partial order $\leq$ on the quotient set $K_f/\equiv$. Let $\text{Core}(K)$ denote the resulting partially ordered set.

In the sequel, it is convenient to consider an isomorphic partially ordered set whose universe is the set of cores of finite structures in $K$. Recall [2, Section 2] that a finite structure is a core if all its endomorphisms are automorphisms. A structure $A$ is a core of a structure $B$ if $A$ is a minimal retract of $B$ (with respect to set inclusion). Simple properties of cores can be found, for example, in [2, Proposition 2.1]. It is easy to see that, in every coset $\mathcal{G}/\equiv$, there exists a unique (up to isomorphism) core. We denote this core by $\text{Core}(\mathcal{G})$. The map defined by the rule $\mathcal{G}/\equiv \mapsto \text{Core}(\mathcal{G})$ is an isomorphism.

Let $K$ be a class of structures. For every $A \in K$, let

$$[K \to A] = \{B \in K : B \to A\}.$$

If there is no ambiguity or $K$ is the class of all structures of a given signature then we write $[\to A]$ instead of $[K \to A]$. For every set $A \subseteq K$, let $[K \to A] = \bigcup_{A \in K} [K \to A]$. If $A$ is a finite set of finite structures then $[K \to A]$ is called a $K$-colour-family. If $|A| = 1$ then the $K$-colour-family $[K \to A]$ is said to be principal. Let $L_0(K)$ denote the partially ordered set of all $K$-colour-families with respect to set inclusion. (If $L_0(K)$ has no greatest element then by $L_0(K)$ we mean the set of all $K$-colour-families with a new greatest element $1_K$).

We recall the definition of operations with structures from [3]. Let $A$ and $B$ be structures and let $(A_i)_{i \in I}$ be a family of structures.

On the disjoint union of the universes of $A$ and $B$, define a structure $A + B$ of signature $\sigma$ as follows: $(a_1, \ldots, a_n) \in r^{A+B}$ if and only if either $(a_1, \ldots, a_n) \in r^A$ or $(a_1, \ldots, a_n) \in r^B$. The resulting structure is called the sum of $A$ and $B$. We have

$$A + B \rightarrow \mathcal{C} \iff A \rightarrow \mathcal{C} \text{ and } B \rightarrow \mathcal{C}$$

for every structure $\mathcal{C}$. A structure $A$ is said to be connected if it cannot be represented in the form $A_1 + A_2$, where $A \rightarrow A_i$, $i = 1, 2$. 


On the Cartesian product of the universes of $A_i$, $i \in I$, define a structure $\prod_{i \in I} A_i$ of signature $\sigma$ as follows: $(a_1, \ldots, a_n) \in r^{\prod_{i \in I} A_i}$ if and only if $(a_1(i), \ldots, a_n(i)) \in r^{A_i}$ for every $i \in I$. The resulting structure is called the product of the family $(A_i)_{i \in I}$. We have

$$\forall C : C \rightarrow \prod_{i \in I} A_i \iff C \rightarrow A_i \text{ for all } i \in I$$

for every structure $C$.

On the set $A^B$ of all functions from $B$ into $A$, define a structure $A^B$ of signature $\sigma$ as follows: $(f_1, \ldots, f_n) \in r^{A^B}$ if and only if $(f_1(b_1), \ldots, f_n(b_n)) \in r^A$ for all $b_1, \ldots, b_n \in B$ with $(b_1, \ldots, b_n) \in r^B$. The resulting structure is called the exponent of $A$ by $B$. We have

$$\forall C : C \rightarrow A^B \iff B \times C \rightarrow A$$

for every structure $C$.

2. LATTICES OF COLOUR-FAMILIES

In this section, we show that partially ordered sets of $K$-colour-families are usually lattices and study their lattice-theoretical properties.

Lemma 1. If $K$ is a class satisfying the condition

$$A \times B \in K \text{ for all finite } A, B \in K$$

then $L_0(K)$ is a distributive lattice with respect to the set-theoretical operations.

Proof. Let $K_1 = [K \rightarrow (A_i)_{i<n}], K_2 = [K \rightarrow (B_j)_{j<m}]$. It is clear that $K_1 \cup K_2$ is a $K$-colour-family, i.e., $K_1 \lor K_2 = K_1 \cup K_2$.

Let $A \in K_1 \cap K_2$. Then $A \in K$ and there exist $i < n$ and $j < m$ such that $A \rightarrow A_i$ and $A \rightarrow B_j$. Hence, $A \rightarrow A_i \times B_j$, cf. (2). By (4), we have $A_i \times B_j \in K$. Conversely, if $A \rightarrow A_i \times B_j$, where $A \in K$, $i < n$, and $j < m$, then we have $A \rightarrow A_i$ and $A \rightarrow B_j$. Hence, $K_1 \cap K_2 = [K \rightarrow (A_i \times B_j)_{i<n, j<m}]$. In view of (4), we have $A_i \times B_j \in K$ for all $i < n$ and $j < m$.  


Remark 2. Another lattice of (principal) colour-families was considered in [4, 5]. In fact, the universe of that lattice is the set of cores, the meet operation coincides with the meet operation in \(L_0(K)\), while the join operation corresponds to the sum of relation structures. That lattice is distributive too.

Let \(L\) be a lattice and let \(a, b \in L\). By a relative pseudocomplement of \(a\) with respect to \(b\) we mean an element \(a * b\) such that

\[a \land x \leq b \iff x \leq a * b\]

for all \(x \in L\). If a relative pseudocomplement exists for every pair of elements of \(L\) then \(L\) is said to be a relatively pseudocomplemented lattice.

Lemma 3. If \(K\) is a class satisfying (4) and the condition

(5) \[A^{B} \in K \text{ for all finite } A, B \in K \text{ with } B \rightarrow A\]

then \(L_0(K)\) is a relatively pseudocomplemented lattice.

**Proof.** Let \(K_1 = [K \rightarrow (A_i)_{i<n}]\) and let \(K_2 = [K \rightarrow (B_j)_{j<m}]\). We introduce the notation \(K^i = [K \rightarrow (A_i)_{j<m}]\). By (5), we have \(B_j^{A_i} \in K\) provided \(A_i \rightarrow B_j\). If \(A_i \rightarrow B_j\) then \(A_i \times C \rightarrow B_j\) for every \(C \in K\). By (3), we obtain \(C \rightarrow B_j^{A_i}\), i.e., \([K \rightarrow B_j^{A_i}] = K\) is the greatest element of \(L_0(K)\).

Therefore, \([K \rightarrow B_j^{A_i}] \in L_0(K)\) for all \(i < n\) and \(j < m\).

By Lemma 1, we have \([K \rightarrow A_i] \cap K^i = [K \rightarrow (B_j^{A_i} \times A_i)_{j<m}]\). It is immediate from (3) that \(B_j^{A_i} \times A_i \rightarrow B_j\). Hence, \([K \rightarrow A_i] \cap K^i \subseteq K_2\). Let \(K_3 = [K \rightarrow (C_k)_{k<l}]\) be a \(K\)-colour-family such that \([K \rightarrow A_i] \cap K_3 \subseteq K_2\). By Lemma 1, we have \([K \rightarrow A_i] \cap K_3 = [K \rightarrow (A_i \times C_k)_{k<l}]\). Therefore, for every \(k < l\), there exists a \(j < m\) such that \(A_i \times C_k \rightarrow B_j\). By definition, \(C_k \rightarrow B_j^{A_i}\). Thus, \(K_3 \subseteq K^i\).

We have proven that \(K_i\) is a pseudocomplement of \([K \rightarrow A_i]\) with respect to \(K_2\). For every distributive lattice \(L\) and elements \(a, b, c \in L\), if \(a * c\) and \(b * c\) exist then so does \((a \lor b) * c\) and \((a \lor b) * c = (a * c) \land (b * c)\), cf. [6, Theorem 9.2.3]. Hence, \(K_1 * K_2 = \bigcap_{i<n} K^i\). \(\blacksquare\)
3. Lattices of Antivarieties

Recall [2] that a $K$-antivariety is a class defined in $K$ by some (possibly, empty) set of anti-identities, i.e., sentences of the form

$$\forall x_1 \ldots \forall x_n (\neg R_1(x) \lor \cdots \lor \neg R_m(x)),$$

where each $R_i(x)$ is an atomic formula. By [2, Theorem 1.2], for every universal Horn class $K$, a subclass $K_0$ is a $K$-antivariety if and only if $K_0 = K \setminus H^{-1}SP_u^*(K')$, where $H^{-1}$, $S$, and $P_u^*$ are operators for taking homomorphic pre-images, substructures, and nontrivial ultraproducts. In particular, each $K$-colour-family is a $K$-antivariety. Let $L(K)$ denote the partially ordered set of all $K$-antivarieties with respect to set inclusion.

**Lemma 4.** For every universal Horn class $K$, the partially ordered set $L(K)$ is a distributive lattice with respect to the set-theoretical operations.

**Proof.** It is clear that $K_1 \land K_2 = K_1 \cap K_2$ and $K_1 \lor K_2 \subseteq K_1 \lor K_2$ for all $K_1, K_2 \in L(K)$. Since $K_1$ and $K_2$ are elementary classes, the class $K_1 \cup K_2$ is elementary too. In particular, $P_u^*(K_1 \cup K_2) \subseteq K_1 \cup K_2$. By [2, Theorem 1.2], we have $K_1 \lor K_2 = K \cap H^{-1}SP_u^*(K_1 \cup K_2)$. Hence, for every $A \in K_1 \lor K_2$, there exists a $B \in K_1 \cup K_2$ such that $A \rightarrow B$. Since $K_1$ and $K_2$ are closed under $H^{-1}S$ in $K$, we obtain $A \in K_1 \cup K_2$ and, consequently, $K_1 \lor K_2 = K_1 \cup K_2$. \qed

The proof of the following lemma is similar to that of [1, Lemma 3.4]

**Lemma 5.** For every universal Horn class $K$ and $K$-antivariety $X$, we have $X = H^{-1}SP_u^*(Core(X)) \cap K$.

If $\sigma$ is a finite signature then the partially ordered set $Core(K)$ and the lattices $L_0(K)$ and $L(K)$ are related as follows.

**Lemma 6.** For every universal Horn class $K$ of finite signature, the lattice $L(K)$ is isomorphic to the ideal lattice $I(L_0(K))$ of the lattice $L_0(K)$ and to the lattice $I_o(Core(K))$ of order ideals of the partially ordered set $Core(K)$.

This lemma generalises [1, Theorem 3.6] to the case of arbitrary relation structures of finite signature. The proof follows the lines of the proof in [1].
**Proof.** Put \( \varphi(K') = \{ A \in L_0(K) : A \subseteq K' \} \) for every \( K' \in \mathcal{L}(K) \) and put \( \psi(J) = \{ A \in L_0(K) : A \in J \} \) for every ideal \( J \) of \( L_0(K) \). It is easy to see that \( \varphi \) is a map from \( \mathcal{L}(K) \) into \( I(L_0(K)) \) and \( \psi \) is a map from \( I(L_0(K)) \) into \( \mathcal{L}(K) \); moreover, both maps are monotone. We prove that \( \varphi = \psi^{-1} \), which implies that \( \varphi \) and \( \psi \) are isomorphisms.

It is clear that \( \varphi \psi(J) \supseteq J \) and \( \psi \varphi(K') \subseteq K' \) for all \( J \in I(L_0(K)) \) and \( K' \in \mathcal{L}(K) \).

Let \( A \in K' \) be a finite structure. Then \([K \rightarrow A] \subseteq K'\) because \( K' \) is a \( K \)-antivariety. We have \( A \in [K \rightarrow A] \subseteq \bigcup \{ A \in L_0(K) : A \subseteq K' \} \). Since the \( K \)-antivariety \( \psi \varphi(K') \) is generated by its finite structures, we obtain \( \psi \varphi(K') = K' \).

Let \( A \in \varphi \psi(J) \), i.e., \( A \in L_0(K) \) and \( A \subseteq \bigvee_{B \in J} B \). Then \( A = [K \rightarrow (A_i)_{i < n}] \), where each finite structure \( A_i \) belongs to \( \bigvee_{B \in J} B \). By [2, Theorem 1.2], for every \( i < n \), there exist a family \( (B_{ij})_{j \in J_i} \subseteq \bigcup_{B \in J} B \) and an ultrafilter \( U_i \) over \( J_i \) such that \( A_i \rightarrow \prod_{j \in J_i} B_{ij} \). Since \( A_i \) is a finite structure of finite signature, from [1, Lemma 3.2] it follows that there exists a \( j(i) \in J_i \) such that \( A_i \rightarrow B_{j(i)} \). Hence, \( A \subseteq [K \rightarrow (B_{j(i)})_{i < n}] \). Since \( B_{j(i)} \subseteq \bigcup_{B \in J} B \), we obtain \( [K \rightarrow B_{j(i)}] \in J \). Since \( J \) is an ideal, we conclude that \( \bigvee_{i < n} [K \rightarrow B_{j(i)}] \in J \). Thus, \( A \in J \) and, consequently, \( \varphi \psi(J) = J \).

For proving the fact that \( \mathcal{L}(K) \) and \( I_0(\text{Core}(K)) \) are isomorphic put

\[ \varphi_0(K') = \{ \text{Core}(A) : A \in K' \}, \quad \psi_0(J) = H^{-1}SP_u^*(J) \cap K. \]

By Lemma 5 and [1, Lemma 3.2], we have \( \varphi_0^{-1} = \psi_0 \). It is clear that \( \varphi_0 \) and \( \psi_0 \) are monotone. Therefore, the lattices \( \mathcal{L}(K) \) and \( I_0(\text{Core}(K)) \) are isomorphic.

**Corollary 7.** For every universal Horn class \( K \) of finite signature, the lattice \( \mathcal{L}(K) \) is relatively pseudocomplemented. For all \( K_1, K_2 \in \mathcal{L}(K) \), the following equality holds: \( K_1 * K_2 = H^{-1}SP_u^*\{ A \in K_i : A \times B \in K_2 \text{ for all } B \in (K_i)_I \} \cap K. \)

**Proof.** By [7, Corollary II.1.4], we have \( I * J = \{ a \in L : a \wedge i \in J \text{ for all } i \in I \} \) for every distributive lattice \( L \) and ideals \( I \) and \( J \) of \( L \). This equality, together with Lemma 6, yields the required assertion.
4. Complexity of lattices of antivarieties

In this section, we introduce the notion of a universal (the most complicated) lattice among the lattices of antivarieties of relation structures of finite signature and give examples of universal lattices.

Let $K$ be a class of structures. By the category $K$ we mean the category whose objects are structures in $K$ and morphisms are homomorphisms. A one-to-one functor $\Phi$ from a category $K_1$ into a category $K_2$ is called a full embedding if, for every morphism $\alpha : \Phi(A) \rightarrow \Phi(B)$ in $K_2$, there exists a morphism $\beta : A \rightarrow B$ in $K_1$ such that $\Phi(\beta) = \alpha$. For more information about categories, the reader is referred to [8]. By $G$ we denote the class (and the category) of undirected loopless graphs.

Lemma 8. Let $K$ be a class of structures and let there exist a full embedding $\Phi : G \rightarrow K$ such that, for every finite graph $G$, the structure $\Phi(G)$ is infinite. Then there exists an embedding $\varphi : \text{Core}(G) \rightarrow \text{Core}(K)$.

Proof. Put $\varphi(G) = \text{Core}(\Phi(G))$ for every $G \in \text{Core}(G)$. It is clear that $\varphi$ is a map from $\text{Core}(G)$ into $\text{Core}(K)$. We show that $\varphi$ is an embedding.

Let $\mathfrak{g} \leq \mathfrak{h}$, where $\mathfrak{g}, \mathfrak{h} \in \text{Core}(G)$, and let $\psi$ be the corresponding homomorphism. Then the composition

$$\varphi(\mathfrak{g}) = \text{Core}(\Phi(\mathfrak{g})) \xrightarrow{\varphi} \Phi(\mathfrak{g}) \xrightarrow{\Phi(\psi)} \Phi(\mathfrak{h}) \xrightarrow{\varphi(\Phi(\psi))} \text{Core}(\Phi(\mathfrak{h})) = \varphi(\mathfrak{h})$$

is a homomorphism from $\varphi(\mathfrak{g})$ into $\varphi(\mathfrak{h})$. Hence, $\varphi(\mathfrak{g}) \leq \varphi(\mathfrak{h})$.

Let $\varphi(\mathfrak{g}) \leq \varphi(\mathfrak{h})$ for some $\mathfrak{g}, \mathfrak{h} \in \text{Core}(G)$ and let $\psi$ be the corresponding homomorphism. Then the composition

$$\Phi(\mathfrak{g}) \xrightarrow{\psi} \text{Core}(\Phi(\mathfrak{g})) = \varphi(\mathfrak{g}) \xrightarrow{\psi} \varphi(\mathfrak{h}) = \text{Core}(\Phi(\mathfrak{h})) \xrightarrow{\varphi(\Phi(\psi))} \Phi(\mathfrak{h})$$

is a homomorphism from $\Phi(\mathfrak{g})$ into $\Phi(\mathfrak{h})$. Denote this homomorphism by $\alpha$. Since $\Phi$ is a full embedding, we have $\alpha = \Phi(\beta)$ for some homomorphism $\beta : \mathfrak{g} \rightarrow \mathfrak{h}$. Hence, $\mathfrak{g} \leq \mathfrak{h}$.

It remains to show that $\varphi$ is a one-to-one map. If $\varphi(\mathfrak{g}) = \varphi(\mathfrak{h})$ then $\varphi(\mathfrak{g}) \leq \varphi(\mathfrak{h})$ and $\varphi(\mathfrak{h}) \leq \varphi(\mathfrak{g})$. By the above, $\mathfrak{g} \leq \mathfrak{h}$ and $\mathfrak{h} \leq \mathfrak{g}$. Hence, $\mathfrak{g} = \mathfrak{h}$.

A category $K$ satisfying the conditions of Lemma 8 is said to be finite-to-finite universal. As is known [9] (see also [10, Theorem 2.10]), the partially ordered set $\text{Core}(G)$ is $\omega$-universal, i.e., each countable partially ordered set
is embeddable into Core(G). By Lemma 8, for every finite-to-finite universal category K, the partially ordered set Core(K) is ω-universal.

Recall that a lattice L is called a factor of a lattice K if L is a homomorphic image of a suitable sublattice of K. We say that L(K) is a universal lattice if, for every universal Horn class K' of relation structures of finite signature, the lattice L(K') is a factor of the lattice L(K).

**Theorem 9.** Let K be a universal Horn class of relation structures of finite signature. If K is a finite-to-finite universal category then L(K) is a universal lattice.

**Proof.** By Lemma 8, for every universal Horn class K' of relation structures of finite signature, there exists an embedding \( \varphi : \text{Core}(K') \rightarrow \text{Core}(K) \).

By Lemma 6, we may consider the lattice of order ideals of the partially ordered set of cores instead of the lattice of antivarieties. For every \( I \subseteq \text{I}_0(\text{Core}(K')) \), put

\[
\psi(I) = \{ \mathcal{H} \in \text{Core}(K) : \mathcal{H} \subseteq \varphi(\mathcal{G}) \text{ for some } \mathcal{G} \in I \}.
\]

It is easy to verify that, for every order ideal I of Core(K'), the set \( \psi(I) \) is an order ideal of Core(K).

We prove that \( \psi \) is one-to-one. Let \( \psi(I) = \psi(J) \). For every \( \mathcal{H} \in I \), we have \( \varphi(\mathcal{H}) \in \psi(I) = \psi(J) \), i.e., there exists an element \( \mathcal{G} \in J \) such that \( \varphi(\mathcal{H}) \leq \varphi(\mathcal{G}) \). Since \( \varphi \) is an embedding, we have \( \mathcal{H} \leq \mathcal{G} \), i.e., \( \mathcal{H} \in J \). We have proven that \( I \subseteq J \). The proof of the converse inclusion is similar.

We prove that \( \psi \) is a join homomorphism. Consequently, the join semilattice of \( \text{I}_0(\text{Core}(K')) \) is embeddable into the join semilattice of \( \text{I}_0(\text{Core}(K)) \). The inclusion \( \psi(I) \lor \psi(J) \leq \psi(I \lor J) \) is obvious. Conversely, let \( \mathcal{H} \in \psi(I \lor J) \). Then there exists a \( \mathcal{G} \in I \lor J \) such that \( \mathcal{H} \leq \varphi(\mathcal{G}) \). Since \( I \lor J = I \lor J \), we obtain \( \mathcal{H} \in \psi(I) \lor \psi(J) = \psi(I) \lor \psi(J) \).

Let L be the sublattice of \( \text{I}_0(\text{Core}(K)) \) generated by the set \( \{ \psi(I) : I \in \text{I}_0(\text{Core}(K')) \} \). Then, for every \( X \in L \), there exist a lattice term \( t(v_0, \ldots, v_{n-1}) \) and order ideals \( J_0, \ldots, J_{n-1} \) of Core(K') such that \( X = t(\psi(J_0), \ldots, \psi(J_{n-1})) \).

We prove that, for every lattice term \( t(v_0, \ldots, v_{n-1}) \) and order ideals \( J_0, \ldots, J_{n-1} \) of Core(K'), the equality

\[
(6) \quad t(\psi(J_0), \ldots, \psi(J_{n-1})) \cap \varphi(\text{Core}(K')) = \varphi(t(J_0, \ldots, J_{n-1}))
\]

holds, where \( \varphi(M) = \{ \varphi(m) : m \in M \} \) for every set M.
We use induction on the length of the term. Let $t(v_0, \ldots, v_{n-1}) = v_i$. Then the right-hand side of (6) is $\varphi(J_i)$ and the left-hand side of (6) is $\psi(J_i) \cap \varphi(\text{Core}(K'))$. It is clear that $\varphi(J_i) \subseteq \psi(J_i) \cap \varphi(\text{Core}(K'))$. Conversely, let $\mathcal{H} \in \psi(J_i) \cap \varphi(\text{Core}(K'))$. Then $\mathcal{H} = \varphi(\mathcal{G})$ for some $\mathcal{G} \in \text{Core}(K')$. Let $J$ be the least ideal of $\text{Core}(K')$ containing $J_i \cup \{\mathcal{G}\}$. We have $\psi(J) = \varphi(J_i) \cup \{A \in \text{Core}(K') : A \subseteq \varphi(\mathcal{G})\}$. Since $\varphi(\mathcal{G}) = \mathcal{H} \in \psi(J_i)$, we obtain $\psi(J) = \psi(J_i)$. Since $\psi$ is a one-to-one map, we have $J = J_i$, i.e., $\mathcal{G} \in J_i$. Therefore, $\mathcal{H} \in \varphi(\mathcal{J}_i) \subseteq \psi(J_i)$.

Assume that $t = t_1 \land t_2$ or $t = t_1 \lor t_2$ for some terms $t_1$ and $t_2$. We introduce the notation

$$Y_i = t_i(\psi(j_0), \ldots, \psi(j_{n-1})), \quad X_i = t_i(j_0, \ldots, j_{n-1}),$$

where $i = 1, 2$. By induction, $Y_i \cap \text{Core}(K') = \varphi(X_i), i = 1, 2$.

If $t = t_1 \land t_2$ then $t(\psi(j_0), \ldots, \psi(j_{n-1})) = Y_1 \cap Y_2, t(j_0, \ldots, j_{n-1}) = X_1 \cap X_2$. By induction, $Y_1 \cap Y_2 \cap \varphi(\text{Core}(K')) = \varphi(X_1 \cap \varphi(X_2)) \supseteq \varphi(X_1 \cap X_2)$. For every $A \in \varphi(X_1) \cap \varphi(X_2)$, there exist $A_i \in X_i, i = 1, 2, \text{such that } A = \varphi(A_1) = \varphi(A_2)$. Since $\varphi$ is a one-to-one map, we obtain $A_1 = A_2 \in X_1 \cap X_2$.

It $t = t_1 \lor t_2$ then $t(\psi(j_0), \ldots, \psi(j_{n-1})) = Y_1 \cup Y_2, t(j_0, \ldots, j_{n-1}) = X_1 \cup X_2$. By induction, $(Y_1 \cup Y_2) \cap \varphi(\text{Core}(K')) = (Y_1 \cap \varphi(\text{Core}(K'))) \cup (Y_2 \cap \varphi(\text{Core}(K'))) = \varphi(X_1) \cup \varphi(X_2) \subseteq \varphi(X_1 \lor X_2)$. The converse inclusion is an easy consequence of the equality $X_1 \lor X_2 = X_1 \cup X_2$.

Since the operations of the lattice of order ideals are the set-theoretical operations, the union and the intersection, from (6) we obtain

$$(7) \quad \varphi^{-1}(t(\psi(j_0), \ldots, \psi(j_{n-1})) \cap \varphi(\text{Core}(K'))) = t(\psi(j_0), \ldots, \psi(j_{n-1})).$$

Let $X = t(\psi(j_0), \ldots, \psi(j_{n-1})) \in L$. Put $\alpha(X) = \varphi^{-1}(X \cap \varphi(\text{Core}(K')))$. It is immediate from (7) that $\alpha$ is a map from $L$ onto $\text{I}_0(\text{Core}(K'))$. By (6), $\alpha$ is a homomorphism.

We present an example showing that the converse to Theorem 9 is not true. Namely, we indicate a quasivariety $K$ of loopless digraphs such that $K$ is not a finite-to-finite universal category but $L(K)$ is a universal lattice.

**Example 10.** Let $\sigma$ consist of one binary relation symbol $r$. Denote by $K$ the quasivariety of structures of the signature $\sigma$ defined by the quasi-identities
∀x∀y(r(x, x) → x ≈ y),

∀x∀y∀z(r(x, y) ∧ r(x, z) → y ≈ z),

∀x∀y∀z(r(y, x) ∧ r(z, x) → y ≈ z).

Let \( C_n, n > 2 \), denote the cycle of length \( n \), i.e., the structure whose universe is \( C_n = \{0, 1, \ldots, n-1\} \) and \( (i, j) \in r^{C_n} \) if and only if \( i + 1 \equiv j \pmod{n} \). It is easy to see that, for every \( n > 2 \), we have \( C_n \subseteq K \).

Let \( \mathbb{P} \) denote the set of prime numbers. Denote by \( a_p, p \in \mathbb{P} \), the \( K \)-colour-family \( [K \to \mathbb{C}_p] \). Let \( L \) be the sublattice of \( L_0(K) \) generated by the elements \( (a_p)_{p \in \mathbb{P}} \).

We show that the distributive lattice \( L \) is freely generated by the set \( (a_p)_{p \in \mathbb{P}} \). Since \( |\mathbb{P}| = \omega \), this means that the free distributive lattice \( F_D(\omega) \) of countable rank is embedded into \( L_0(K) \). We use [7, Theorem II.2.3]. It suffices to verify that, for all finite nonempty subsets \( I, J \subseteq \mathbb{P} \), from \( \bigwedge_{i \in I} a_i \leq \bigvee_{j \in J} a_j \) it follows that \( I \cap J \neq \emptyset \).

Let \( I \) and \( J \) be finite and nonempty. Assume that \( \bigwedge_{i \in I} a_i \leq \bigvee_{j \in J} a_j \). By Lemma 1, we have \( \bigwedge_{i \in I} a_i = [K \to \prod_{i \in I} \mathbb{C}_i] \) and \( \bigvee_{j \in J} a_j = [K \to \mathbb{C}_j]_{j \in J} \). Let \( k = \prod_{i \in I} i \). It is easy to see that \( \prod_{i \in I} \mathbb{C}_i \cong \mathbb{C}_k \) (cf., for example, [11]). Since \( \bigwedge_{i \in I} a_i \leq \bigvee_{j \in J} a_j \), there exists a prime \( j \in J \) with \( \mathbb{C}_k \subseteq [K \to \mathbb{C}_j] \). We have \( \mathbb{C}_k \to \mathbb{C}_j \) if and only if \( j \) divides \( k \) (cf., for example, [12]). Since \( j \) is prime and \( k \) is a product of distinct primes, we conclude that \( j \in I \). Thus, \( I \cap J \neq \emptyset \).

We show that the ideal lattice \( I(F_D(\omega)) \) of the free distributive lattice of countable rank is embeddable into \( L(K) \). Let \( L \) and \( K \) be distributive lattices and let \( \varphi : L \to K \) be an embedding. Define a map \( \psi : I(L) \to I(K) \) by the following rule: \( \psi(I) \) is the ideal of \( K \) generated by \( \varphi(I) \). Using the definition of an ideal generated by a set, we easily find that \( \psi \) is an embedding. In particular, \( I(F_D(\omega)) \) is embeddable into \( I(L_0(K)) \). The latter lattice is isomorphic to \( L(K) \) in view of Lemma 6.

We show that the lattice \( L(G) \) of antivarieties of undirected loopless graphs is a homomorphic image of the lattice \( I(F_D(\omega)) \). Since \( L_0(G) \) is a countable distributive lattice, there exists a homomorphism from \( F_D(\omega) \) onto \( L_0(G) \). As above, this homomorphism induces a homomorphism between the corresponding ideal lattices. It remains to use Lemma 6.
Therefore, \( L(G) \) is a factor of \( L(K) \). We conclude that \( L(K) \) is a universal lattice. The class of rigid objects in the category \( K \) consists of trivial structures and finite directed chains only (cf. [8, Exercise IV.1.6]). Therefore, the category \( K \) is not universal and, consequently, is not finite-to-finite universal.

5. Irredundant meet decompositions in lattices of colour-families

Recall that \( G \) denotes the universal Horn class and the category of undirected loopless graphs. The study of the lattice \( L_0(G) \) was initiated in [1]. It was proven that this lattice possesses neither completely join irreducible nor completely meet irreducible nonzero elements. A simple description for join irreducible colour-families was found. The question on meet irreducible elements turned to be closely connected with a well-known problem in the graph theory, Hedetniemi’s conjecture [13].

Here, we consider meet decompositions of \( K \)-colour-families with the help of Lemma 3, which says that the lattice of \( K \)-colour-families is relatively pseudocomplemented. A similar approach was first used in [4].

We present necessary definitions. By a \textit{meet decomposition} of an element \( x \in L \), where \( L \) is an arbitrary lattice, we mean a representation

\[
x = \bigwedge_{i \in I} m_i,
\]

where \( (m_i)_{i \in I} \) is a family of meet irreducible elements, i.e., for each \( i \in I \), we have \( m_i \neq 1 \) and from \( m_i = a \wedge b \) it follows that either \( m_i = a \) or \( m_i = b \). A meet decomposition (8) is \textit{irredundant} if \( x < \bigwedge_{i \in J} m_i \) for every proper subset \( J \subseteq I \). For distributive relatively pseudocomplemented lattices, the following criterion for meet irreducibility of elements is known [14].

\textbf{Proposition 11.} Let \( L \) be a distributive relatively pseudocomplemented lattice and let \( m \in L \). The element \( m \) is meet irreducible if and only if \( x \ast m = m \) for every \( x \in L \) with \( x \not\leq m \).

Throughout this section, we assume that \( L \) is an arbitrary distributive relatively pseudocomplemented lattice. Let \( \vee, \wedge, \ast \) denote the operations of \( L \). For every \( x \in L \), let \( \text{Reg}(x) = \{ y \ast x : y \in L \} \), i.e., let \( \text{Reg}(x) \) denote the
set of regular elements of the principal filter \([x]\) of \(L\). For all \(u, v \in \text{Reg}(x)\), put
\[
u + v = ((u \lor v) \ast x) \ast x, \quad u \cdot v = u \land v, \quad 0 = x, \quad 1 = 1_L, \quad u' = u \ast x.
\]
The set \(\text{Reg}(x)\) with the operations \(\cdot\), \(\ast\), and \(\cdot\) and constants \(1\) and \(0\) is a Boolean algebra; moreover, the map \(r\) from \(L\) to \(\text{Reg}(x)\) defined by the rule \(r(y) = (y \ast x) \ast x\) is a homomorphism between Heyting algebras [6, Theorem 8.4.3].

We mention the following relationship between meet irreducible elements of \(L\) and dual atoms of \(\text{Reg}(x)\) [4, Theorem 6].

**Proposition 12.** Let \(x, y \in L\) and let \(x < y\). The element \(y\) is a dual atom of the Boolean algebra \(\text{Reg}(x)\) if and only if \(y \ast x > x\) and \(y\) is meet irreducible in \(L\).

Recall [15] that a Boolean algebra \(A\) is **atomic** if, for every nonzero element \(a \in A\), there exists an atom \(b\) such that \(b \leq a\). An element \(a\) is said to be **atomless** if \(a \neq 0\) and there is no atom \(b\) such that \(b \leq a\).

**Theorem 13.** Let \(L\) be a distributive pseudocomplemented lattice and let \(a \in L\). The element \(a\) admits an irredundant meet decomposition in \(L\) if and only if the Boolean algebra \(\text{Reg}(a)\) is atomic.

**Proof.** Let \(a = \bigwedge_{i \in I} m_i\) be an irredundant meet decomposition. We prove that \(m_i \ast a > a\) for every \(i \in I\). In view of Proposition 12, this means that each \(m_i\), \(i \in I\), is a dual atom of \(\text{Reg}(a)\). Since \(a = \bigwedge_{i \in I} m_i\), we have \(m_i \ast a = m_i \ast (\bigwedge_{i \in I} m_i) = \bigwedge_{i \neq j} m_i \ast m_j = \bigwedge_{j \neq i} m_j\) (cf. [16, IV.7.2 (8)]). Since the meet decomposition is irredundant, we have \(m_i \ast a = \bigwedge_{j \neq i} m_j > a\).

Assume that there exists an atomless element \(b \in \text{Reg}(a)\). Since the complement of a dual atom is an atom, we conclude that \(m_i \ast a \not\leq b\) for all \(i \in I\). Hence, \(b \land (m_i \ast a) = a\) for all \(i \in I\). By the definition of a relative pseudocomplement, we have \(b \leq (m_i \ast a) \ast a = m_i\) for all \(i \in I\). Therefore, \(b \leq \bigwedge_{i \in I} m_i = a\). This proves that \(\text{Reg}(a)\) possesses no atomless element, i.e., \(\text{Reg}(a)\) is an atomic Boolean algebra.

Conversely, assume that \(\text{Reg}(a)\) is an atomic Boolean algebra. Let \((a_i)_{i \in I}\) be the set of atoms. If \(|\text{Reg}(a)| = 2\) then the element \(a\) is meet irreducible in view of Proposition 11. In the sequel, we assume that \(|\text{Reg}(a)| > 2\). We denote \(m_i = a_i \ast a, i \in I\). For every \(i \in I\), the element \(m_i\) is a dual atom of \(\text{Reg}(a)\); moreover, each dual atom is of the form \(m_i, i \in I\).
By Proposition 12, we have $m_i \ast a > a$ and $m_i$ is meet irreducible for every $i \in I$. It remains to show that $a = \bigwedge_{i \in I} m_i$ (in the lattice $L$). It is clear that $a$ is a lower bound for $(m_i)_{i \in I}$. If $a$ is not the greatest lower bound then there exists a lower bound $b_0 \in L$ for $(m_i)_{i \in I}$ such that $b_0 \not\leq a$. Consider the element $(b_0 \ast a) \ast a \in \text{Reg}(a)$. Since $b_0 \leq m_i$, we conclude that $(b_0 \ast a) \ast a \leq (m_i \ast a) \ast a = m_i$, $i \in I$. Hence, $b = (b_0 \ast a) \ast a \in \text{Reg}(a)$ is a lower bound for $(m_i)_{i \in I}$. Since $b_0 \not\leq a$ and $b_0 \leq b$, we find that $b \neq a$, i.e., $b > a$. Since $\text{Reg}(a)$ is an atomic Boolean algebra, there exists an atom $a_j$ such that $a_j \leq b$. Passing to the complements, we find that $b' \leq m_j$. Since $b \leq m_i$ for all $i \in I$, we obtain $b + b' = 1 \leq m_j + m_j = m_j < 1$, a contradiction.

Similar questions for undirected loopless graphs were considered in [17], where the notion of the level of nonmultiplicativity of a graph was introduced. In our terminology, the level of nonmultiplicativity of a graph $\mathcal{G}$ is the number of dual atoms of the Boolean algebra $\text{Reg}([\mathcal{G} \to \mathcal{G}])$. In [17], the following conjecture is stated: The level of nonmultiplicativity of each finite graph is finite. In connection with Theorem 13, we formulate the following

**Problem 14.** Let $K$ be a universal Horn class of relation structures of finite signature. Is it true that, for every $K$-colour-family $A$, the following conditions are equivalent:

1. there exists an irredundant meet decomposition $A = \bigwedge_{i \in I} M_i$,
2. there exists a finite meet decomposition $A = \bigwedge_{i < n} M_i$, $n < \omega$,
3. the Boolean algebra $\text{Reg}(A)$ is finite?

In the next section, we find a connection between this problem and existence of independent bases for anti-identities.

6. **Anti-identities of finite structures**

Recall that a set $\Sigma$ of anti-identities is a basis for anti-identities of a class $K$ if $K$ is the class of structures in which all anti-identities of $\Sigma$ are valid, i.e., $K = \text{Mod}(\Sigma)$. By a basis for anti-identities of a structure $A$ we mean a basis for anti-identities of the antivariety generated by $A$. A basis $\Sigma$ is said to be independent if, for every $\varphi \in \Sigma$, the proper inclusion $\text{Mod}(\Sigma) \not\subseteq \text{Mod}(\Sigma \setminus \{\varphi\})$ holds.
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A structure \( \mathcal{A} \) is said to be weakly atomic compact if every locally consistent set of atomic formulas is consistent in \( \mathcal{A} \).

We reduce the question on existence of an independent basis for anti-identities of a finite relation structure of finite signature to Problem 14. Let \( \mathcal{A} \) be a finite relation structure of finite signature and let \( \Sigma = (\varphi_i)_{i \in I} \) be an independent basis for anti-identities of \( \mathcal{A} \). With each anti-identity \( \varphi_i \), \( i \in I \), we associate a finitely presented structure \( \mathcal{B}_i \) as follows:

Let \( \varphi_i \equiv \forall \overline{x} (\neg \psi_1(\overline{x}) \lor \cdots \lor \neg \psi_n(\overline{x})) \); then \( \mathcal{B}_i \) is the structure defined by generators \( \overline{x} \) and relations \( \psi_1(\overline{x}), \ldots, \psi_n(\overline{x}) \).

It is easy to see that the antivariety defined by \( \Sigma \) coincides with the class

\[
\bigcap_{i \in I} [\mathcal{B}_i \rightarrow] = \{ \mathcal{B} : \mathcal{B}_i \rightarrow \mathcal{B} \text{ for all } i \in I \}.
\]

Since \( \Sigma \) is an independent basis, we have \( \mathcal{B}_i \rightarrow \mathcal{B}_j \) if and only if \( i = j \).

**Lemma 15.** The following two conditions are equivalent:

1. \( [\rightarrow \mathcal{A}] = \bigcap_{i \in I} [\mathcal{B}_i \rightarrow] \),
2. there exists a family of finite structures \( (\mathcal{A}_i)_{i \in I} \) such that \( [\mathcal{B}_i \rightarrow] = [\rightarrow \mathcal{A}_i] \) for all \( i \in I \) and \( [\rightarrow \mathcal{A}] = [\rightarrow \prod_{i \in I} \mathcal{A}_i] \).

**Proof.** It is clear that (2) implies (1). Indeed, if such a family exists then, for every structure \( \mathcal{C} \), we have

\[
\mathcal{C} \in [\rightarrow \mathcal{A}] = \left[ \rightarrow \prod_{i \in I} \mathcal{A}_i \right] \iff \mathcal{C} \rightarrow \mathcal{A}_i \text{ for all } i \in I \iff \mathcal{C} \rightarrow \mathcal{B}_i \text{ for all } i \in I \iff \mathcal{C} \in \bigcap_{i \in I} [\mathcal{B}_i \rightarrow].
\]

We prove that (1) implies (2).

Notice that each structure \( \mathcal{B}_i \), \( i \in I \), is connected. Assume the contrary, i.e., let there exist an element \( i \in I \) such that \( \mathcal{B}_i \) is not connected. Then \( \mathcal{B}_i = \mathcal{B}_i^1 + \mathcal{B}_i^2 \) for some structures \( \mathcal{B}_i^k \) with \( \mathcal{B}_i \rightarrow \mathcal{B}_i^k \), \( k = 1, 2 \). Since \( \mathcal{B}_j \rightarrow \mathcal{B}_i \) provided \( j \neq i \), we have \( \mathcal{B}_i^k \in [\mathcal{B}_j \rightarrow] \) for all \( j \neq i \) and \( k = 1, 2 \). Therefore, \( \mathcal{B}_i^k \in \bigcap_{i \in I} [\mathcal{B}_i \rightarrow] = [\rightarrow \mathcal{A}] \), \( k = 1, 2 \). Thus, \( \mathcal{B}_i \in [\rightarrow \mathcal{A}] \subseteq [\mathcal{B}_i \rightarrow] \), which is a contradiction.
For an arbitrary $i \in I$, consider the interval $[A_i, A_i + B_i]$ of the partially ordered set of cores. If there exists a core $C$ such that $A_i \not\leq C \not< A_i + B_i$ and $B_i \not< C \not< A_i$, then, by (1), we obtain $C \not< \bigcap_{i \in I} [B_i \not< i]$. Hence, there exists a $j \in I$ such that $B_j \not< C \not< A_i + B_i$. It is easy to see that $i \neq j$. Since $B_j \not< C \not< A_i + B_i$ and $B_j$ is connected, we conclude that $B_j \not< B_i$, where $i \neq j$. Since $\Sigma$ is an independent basis, we arrive at a contradiction. Thus, $A_i + B_i$ covers $A_i$ in the partially ordered set of cores (in symbols: $A_i \not< A_i + B_i$). Since this is a distributive lattice (cf. Remark 2), we conclude that $A_i \not< B_i$. We denote $C_i = A_i + B_i$, $i \in I$. By (2) and (3), we obtain $C_i \not< \bigcap_{i \in I} [B_i \not< i]$. We deduce

$$D \in \bigcap_{i \in I} [B_i \not< i] \iff D \in [B_i \not< i] \text{ for all } i \in I \iff D \in [-A_i] \text{ for all } i \in I \iff D \in \left[\leftarrow \prod_{i \in I} A_i \right].$$

Since $\bigcap_{i \in I} [B_i \not< i] = [-A]$, we obtain $[-A] = \left[\leftarrow \prod_{i \in I} A_i \right]$. Moreover, if the structure $A$ has no trivial substructure then the structure $\prod_{i \in I} A_i$ has no trivial substructure either.

In the sequel, we assume that (equivalent) conditions (1) and (2) of Lemma 15 are satisfied. Without loss of generality, we may assume that $A$ is a core. Since $A$ is finite, the class $[-A]$ is elementary. By [2, Proposition 2.2], the structure $\prod_{i \in I} A_i$ is weakly atomic compact and $[-\prod_{i \in I} A_i]$ is the antivariety generated by $\prod_{i \in I} A_i$. In view of [2, Corollary 2.6], there exists a unique (up to isomorphism) core $A^*$ of $\prod_{i \in I} A_i$; moreover, the antivarieties generated by $A^*$ and $\prod_{i \in I} A_i$ coincide. Therefore, the antivarieties generated by $A$ and $A^*$ coincide. By [2, Corollary 2.5], the structures $A$ and $A^*$ are isomorphic.

Since $A^*$ is a finite structure of finite signature, there exists a finite subset $F \subseteq I$ such that $A^*$ is embeddable into $\prod_{i \in F} A_i$.

We suggest the following
Conjecture 16. The equality \([-\rightarrow \mathcal{A}] = \bigwedge_{i \in F'} \mathcal{A}_i\) holds for some finite subset \(F' \subseteq I\) with \(F \subseteq F'\).

If this conjecture is true then \(\Sigma\) is a finite basis, which means that every finite relation structure of finite signature having no finite basis for its anti-identities possesses no independent basis for its anti-identities.

We return to Problem 14. Let \(\mathbf{K}_i\) be the principal colour-family generated by \(\mathcal{A}_i, i \in I\). Then

\[\bigwedge_{i \in I} \mathbf{K}_i\]

is an irredundant meet decomposition of \([-\rightarrow \mathcal{A}]\) (in the lattice of colour-families). Therefore, if the answer to the question in Problem 14 is positive then Conjecture 16 is true.

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