THE TABLE OF CHARACTERS
OF SOME QUASIGROUPS

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Abstract

It is known that $(\mathbb{Z}_n, -_n)$ are examples of entropic quasigroups which are not groups. In this paper we describe the table of characters for quasigroups $(\mathbb{Z}_n, -_n)$.

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1. Introduction

The theory of characters of finite quasigroup has been already considered by J.D.H. Smith in [3].

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A quasigroup \((Q, \cdot)\) is a set \(Q\) equipped with a binary multiplication operation denoted by \(\cdot\) or juxtaposition of the two arguments, in which specification of any two of \(x, y, z\) in the equation \(x \cdot y = z\) determines the third uniquely.

A quasigroup \((Q, \cdot)\) is called entropic if

\[
(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)
\]

for all \(x, y, z, t \in Q\).

Let \((Q, \cdot)\) be a finite quasigroup. Now we describe how to obtain the character table of \(Q\) (see [3], Chapter 5).

Let \(R: Q \to Q!; x \mapsto R(x)\) and \(L: Q \to Q!; x \mapsto L(x)\), where \(R(x)(q) = q \cdot x\) and \(L(x)(q) = x \cdot q\). Then the subgroup \(G = \text{Mit}(Q, \cdot)\) of \(Q!\) generated by the union \(R(Q) \cup L(Q)\) is called the multiplication group of the quasigroup \((Q, \cdot)\).

The group \(G\) acts onto \(Q \times Q\) in the following way:

\[
g: Q \times Q \to Q \times Q; \quad (x, y) \mapsto (g(x), g(y)).
\]

The orbits \(\{C_1, \ldots, C_s\}\) of \(G\) on \(Q \times Q\) under this action are called the conjugacy classes of \(Q\).

We consider the incidence matrix \(a_i\) of the conjugacy class \(C_i\). This is a 0–1-matrix having 1 as its \(xy\)-component if \((x, y) \in C_i\) and 0 otherwise.

The space \(\mathbb{C}Q\) can be decomposed as a direct sum of subspaces \(E_j\) such that

\[
(a) \quad \forall 1 \leq i \leq s, \exists i_{ij} \in \mathbb{C}E_j(a_i - \xi_{ij}I) = \{0\};
\]

\[
(b) \quad \forall j \neq k, \exists i \in \mathbb{C}E_j \neq \xi_{ik};
\]

\[
(c) \quad E_1 = \mathbb{C} \left( \sum_{q \in \mathbb{Q}} q \right).
\]

To get \((a)\) and \((b)\), decompose \(\mathbb{C}Q\) into \(a_1\)-eigenspaces, then decompose each of these into \(a_2\)-eigenspaces, and so on. In the case of quasigroup \((\mathbb{Z}_n, \cdot)\) it is enough to end this process with \(a_2\)-eigenspaces. Let \(e_j: \mathbb{C}Q \to E_j\) be the projection onto \(E_j\). Define \((s \times s)\)-matrix \(\Xi = (\xi_{ij})\) by \(a_i = \sum_{j=1}^{s} \xi_{ij}e_j\).
Finally the *character table* of the quasigroup $Q$ is the complex $(s \times s)$ matrix $\Psi$ with components

$$\psi_{il} = (f_i)^{-1} \xi_{li} n_l^{-1},$$

for $i, l = 1, \ldots, s$, where $f_i = \text{dim}_C E_i$ and $n_l = |C_l|/|Q|$.  

For more details see [1, 3, 5].  

In this paper we find the character tables of quasigroups $(Z_n, -n)$. If $i, j \in \mathbb{Z}_n$ then

$$i - n j = \begin{cases} 
i - j \text{ for } i \geq j \\ n + i - j \text{ for } i < j \end{cases}.$$  

Every quasigroup $(Z_n, -n)$ has the following conjugacy classes:

$$C_i = \{(k, t) \in \mathbb{Z}_n^2 : |k - t| = i - 1 \text{ or } |k - t| = n - i + 1\}$$

for $i = 1, \ldots, \left[\frac{n}{2}\right]$. One can check that $|C_j| = n$ if $j = 1$ or $(j = \frac{n}{2} + 1$ and $2|n)$ and $|C_j| = 2n$ otherwise.

This is a „road map” through the lemmas in this paper:
2. Notations

For $n \in \mathbb{N}$, $0 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $m \in \mathbb{N}$ let

$$x_{n,m} = \begin{cases} 2\cos \frac{2m\pi}{n} & \text{if } 2|n \\ (-1)^m 2\cos \frac{m\pi}{n} & \text{otherwise.} \end{cases}$$

For $n \in \mathbb{N}$ define the function $g_n : \mathbb{Z} \rightarrow \{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\}$ in the following way $g_n(x) = \text{dist}(x, n\mathbb{Z})$. Let $a_i$ be the incidence matrix of the conjugacy class $C_i$. This is a $0-1$-matrix having $1$ as its $xy$-component if $(x, y) \in C_i$ and $0$ otherwise. Let $w_n$ be the characteristic polynomial of $a_2$.

3. Main theorem

In this section we prove a recursive formula for the characteristic polynomial of the matrix $a_2$. Before that we give and prove necessary lemmas.

**Lemma 1.** For every $n \geq 3$ we have

$$w_{n+2}(x) = -xw_{n+1}(x) - w_n(x) + (-1)^n(2x - 4).$$

**Proof.** Let $v_n = (b_{ij})_{1 \leq i, j \leq n}$ be the matrix such that

$$b_{ij} = \begin{cases} 0 & \text{for } |i - j| \geq 2 \\ 1 & \text{for } |i - j| = 1 \\ -x & \text{for } i = j. \end{cases}$$

By Laplace’s expansion of the determinant along 1 column we have

(1) \hspace{1cm} v_n(x) = -xv_{n-1} - v_{n-2}(x).

Using again Laplace’s formula to expand the determinant along 1 column and 1 row we have
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(2)

\[ w_n(x) = -xv_{n-1}(x) - (v_{n-2} + (-1)^n) + (-1)^{n+1}(1 + (-1)^nv_{n-2}(x)) \]

Now we obtain

\[ w_{n+2}(x) = -xv_{n+1} + 2v_n + 2 \cdot (-1)^{n+1} = -x(-xv_n(x) - v_{n-1}(x)) \]

\[ -2v_n(x) + 2 \cdot (-1)^{n+1} = v_n(x)(x^2 - 2) + xv_{n-1}(x) + 2 \cdot (-1)^{n+1} \]

\[ = x^2v_n(x) + 2xv_{n-1}(x) - 2x(-1)^n + 2v_n(x) - xv_{n-1}(x) + 2x(-1)^n \]

\[ = -xw_{n+1}(x) \]

\[ = -xw_{n+1}(x) + xv_{n-1}(x) + 2v_{n-2}(x) - 2(-1)^{n+1} \]

\[ = -w_n(x) \]

\[ = -2xv_{n-1}(x) - 2v_n(x) - 2v_{n-2}(x) + 4(-1)^{n+1} + 2x(-1)^n \]

\[ = 0 \text{ by (1)} \]

\[ = -xw_{n+1}(x) - w_n(x) + (-1)^n(2x - 4). \]

Let \( u_n(x) \) be a polynomial such that \( u_{2n+2}(x) = u_{2n+1}(x) - u_{2n}(x) \), \( u_{2n+1}(x) = (x + 2)u_{2n}(x) - u_{2n-1}(x) \) and \( u_1(x) = u_2(x) = 1 \).

Lemma 2. For every \( n \in \mathbb{N} \) we have

(a) \( (x + 2)u_{2n}(x)u_{2n+1}(x) = u_{2n+1}(x) + (x + 2)u_{2n}(x) - 1 \),

(b) \( (x + 2)u_{2n+2}(x)u_{2n+1}(x) = u_{2n+1}(x) + (x + 2)u_{2n+2}(x) - 1 \).
Proof. For \( n = 1 \) it is clear. Assume that lemma is true for \( n \). We prove this lemma for \( n + 1 \).

\[
    u_{2n+3}^2(x) + (x + 2)u_{2n+2}^2(x) - 1 = ((x + 2)u_{2n+2}(x) - u_{2n+1}(x))u_{2n+3}(x) + (x + 2)u_{2n+2}(x) - 1 = (x + 2)u_{2n+2}(x)u_{2n+3}(x) - u_{2n+1}(x)((x + 2)u_{2n+2}(x) - 1)
\]

by \((b)\).

\[
    -u_{2n+1}(x) + (x + 2)u_{2n+2}(x) - 1 = (x + 2)u_{2n+2}(x)u_{2n+3}(x)
\]

\[-(u_{2n+1}^2(x) + (x + 2)u_{2n+2}(x) - 1) + u_{2n+1}^2(x) + (x + 2)u_{2n+2}^2(x) - 1
\]

\[= (x + 2)u_{2n+2}(x)u_{2n+3}(x),
\]

hence \((a)\) is true for \( n + 1 \).

\[
    u_{2n+3}^2(x) + (x + 2)u_{2n+4}^2(x) - 1 = u_{2n+3}^2(x) + (x + 2)u_{2n+4}(x)u_{2n+3}(x)
\]

\[-u_{2n+2}(x) - 1 = u_{2n+3}^2(x) + (x + 2)u_{2n+4}(x)u_{2n+3}(x)
\]

\[-(x + 2)u_{2n+4}(x)u_{2n+3}(x)
\]

\[= u_{2n+3}^2(x) + (x + 2)u_{2n+4}(x)u_{2n+3}(x)
\]

\[-(x + 2)(u_{2n+3}(x) - u_{2n+2}(x))u_{2n+2}(x) - 1
\]

\[= u_{2n+3}^2(x) + (x + 2)u_{2n+3}(x) - 1 + (x + 2)u_{2n+4}(x)u_{2n+3}(x)
\]

\[-(x + 2)u_{2n+3}(x)u_{2n+2}(x) = (x + 2)u_{2n+4}(x)u_{2n+3}(x)
\]

\[+ (x + 2)u_{2n+4}(x)u_{2n+3}(x) - (x + 2)u_{2n+3}(x)u_{2n+2}(x)
\]

\[= (x + 2)u_{2n+4}(x)u_{2n+3}(x)
\]

so we obtain \((b)\) for \( n + 1 \).
Now we pass to the lemma expressing polynomial $w_n$ by $u_n$.

**Lemma 3.** For every $n \geq 1$

(a) $w_{2n+1}(x) = (2 - x)u_{2n+1}^2(x),$

(b) $w_{2n}(x) = (x^2 - 4)u_{2n}^2(x).$

**Proof.** For $n = 2$ it is obvious. Assume that lemma is true for $n$. We prove lemma for $n + 1$. Using Lemma 1 and Lemma 2 we have

\[
\begin{align*}
w_{2n+2}(x) \overset{L1}{=} & \ -xw_{2n+1}(x) - w_{2n}(x) + 2x - 4 = -x(2 - x)u_{2n+1}^2(x) \\
& - (x^2 - 4)u_{2n}^2(x) + 2x - 4 \\
\overset{L2a}{=} & \ (x^2 - 2x)u_{2n+1}^2(x) - (x^2 - 4)u_{2n}^2(x) + 2x - 4 \\
& + (2x - 4)(u_{2n+1}^2(x) - (x + 2)u_{2n}(x)u_{2n+1}(x) - 1 + u_{2n}^2(x)(x+2)) \\
& = (x^2 - 4)u_{2n+1}^2(x) + (x^2 - 4)u_{2n}^2(x) - 2(x^2 - 4)u_{2n}(x)u_{2n+1}(x) \\
& = (x^2 - 4)(u_{2n+1}^2(x)u_{2n}^2(x) - 2u_{2n}(x)u_{2n+1}(x)) \\
& = (x^2 - 4)(u_{2n+1}(x) - u_{2n}(x))^2 = (x^2 - 4)u_{2n+2}^2(x)
\end{align*}
\]

so we obtain (b) for $n + 1$. By Lemma 1 and 2 and (b) for $n + 1$ we have

\[
(2 - x)u_{2n+3}^2(x) = (2 - x)((x + 2)u_{2n+2}(x) - u_{2n+1}(x))^2
\]

\[
\overset{L2b}{=} \ (2 - x)((x + 2)u_{2n+2}(x) - u_{2n+1}(x))^2 \\
& + (2x - 4)((x + 2)u_{2n+2}^2(x) + u_{2n+1}^2(x) \\
& - 1 - (x + 2)u_{2n+2}(x)u_{2n+1}(x)) =
\]


\[
(x - 2)(- (x + 2)^2 u_{2n+2}^2(x) \\
+ 2(x + 2)u_{2n+1}(x)u_{2n+2}(x) - u_{2n+1}^2(x) \\
+ 2(x + 2)u_{2n+2}^2(x) + 2u_{2n+1}^2(x) - 2 \\
- 2(x + 2)u_{2n+2}(x)u_{2n+1}(x)
\]

\[
= (x - 2)(u_{2n+2}^2(x)(-x^2 - 2x) + u_{2n+1}^2 - 2)
\]

\[
= -x(x^2 - 4)u_{2n+2}^2(x) - (2 - x)u_{2n+1}^2(x) - 2x + 4
\]

\[
= xw_{2n+2}(x) - w_{2n+1}(x) - 2x + 4 \overset{L_1}{=} w_{2n+3}(x)
\]

hence (α) is true for \( n + 1 \).

\[\Box\]

**Lemma 4.** Let \( n \in \mathbb{N} \) and \( 0 \leq j, k \leq \left[ \frac{n}{2} \right] \). Then

\[
x_{n,j} \cdot x_{n,k} = x_{n,\lfloor k - j \rfloor} + x_{n,g_n(k+j)}.
\]

**Proof.** Consider the following cases:

1. \( n \) is odd and \( j + k \leq \left[ \frac{n}{2} \right] \). Then

\[
x_{n,j} \cdot x_{n,k} = 2 \cos \left( \frac{2j}{n} \right) 2 \cos \left( \frac{2k}{n} \right)
\]

\[
= 2 \left( \cos \left( \frac{2(j - k)}{n} \right) + \cos \left( \frac{2(j + k)}{n} \right) \right)
\]

\[
= x_{n,\lfloor k - j \rfloor} + x_{n,g_n(k+j)}.
\]
2. \( n \) is odd and \( j + k > \left[ \frac{n}{2} \right] \). Then \( g_n(j + k) = n - (j + k) \) and

\[
x_{n,j} \cdot x_{n,k} = 2 \cos \left( \frac{2j\pi}{n} \right) 2 \cos \left( \frac{2k\pi}{n} \right)
\]

\[
= 2 \left( \cos \left( \frac{2(j - k)\pi}{n} \right) + \cos \left( \frac{2(j + k)\pi}{n} \right) \right)
\]

\[
= 2 \left( \cos \left( \frac{2(j - k)\pi}{n} \right) + \cos \left( 2\pi - \frac{2(j + k)\pi}{n} \right) \right)
\]

\[
= 2 \cos \left( \frac{2(j - k)\pi}{n} \right) + \cos \left( \frac{2(n - (j + k))\pi}{n} \right) = x_{n,|k-j|} + x_{n,g_n(k+j)}.
\]

3. \( n \) is even and \( j + k \leq \left[ \frac{n}{2} \right] \). Then

\[
x_{n,j} \cdot x_{n,k} = (-1)^j 2 \cos \left( \frac{j\pi}{n} \right) (-1)^k 2 \cos \left( \frac{k\pi}{n} \right)
\]

\[
= (-1)^{j+k} 2 \left( \cos \left( \frac{(j-k)\pi}{n} \right) + \cos \left( \frac{(j+k)\pi}{n} \right) \right) = x_{n,|k-j|} + x_{n,g_n(j+k)}.
\]

4. \( n \) is even and \( j + k > \left[ \frac{n}{2} \right] \). Then

\[
x_{n,j} \cdot x_{n,k} = (-1)^j 2 \cos \left( \frac{j\pi}{n} \right) (-1)^k 2 \cos \left( \frac{k\pi}{n} \right)
\]

\[
= (-1)^{j+k} 2 \left( \cos \left( \frac{(j-k)\pi}{n} \right) + \cos \left( \frac{(j+k)\pi}{n} \right) \right)
\]

\[
= (-1)^{k-j} 2 \cos \left( \frac{(j-k)\pi}{n} \right) + (-1)^{j+k} 2(-1) \cos \left( \pi - \frac{(j+k)\pi}{n} \right)
\]

\[
= (-1)^{k-j} 2 \cos \left( \frac{(j-k)\pi}{n} \right) + (-1)^{n-(j+k)} 2 \cos \left( \pi - \frac{(j+k)\pi}{n} \right)
\]

\[
= x_{n,|k-j|} + x_{n,g_n(j+k)}.
\]
Lemma 5. Let \( n \in \mathbb{N} \), \( y \in \mathbb{Z} \) and \( j \in \{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}. \) Then
\[
\{g_n(j + g_n(y)), g_n(y) - j\} = \{g_n(y - j), g_n(y + j)\}.
\]

Proof. There exists \( k \in \mathbb{Z} \) such that \( kn \leq y \leq kn + n \). Let us consider the following cases:

1. If \( y - kn \leq \left\lfloor \frac{n}{2} \right\rfloor \) then \( g_n(y) = y - kn \) and
\[
g_n(y + j) = \text{dist}(y + j, n\mathbb{Z}) = \text{dist}(y - kn + j, n\mathbb{Z})
\]
   \[= \text{dist}(g_n(y) + j, n\mathbb{Z}) = g_n(g_n(y) + j)\]
   and
\[
g_n(y - j) = \text{dist}(y - j, n\mathbb{Z}) = \text{dist}(y - kn - j, n\mathbb{Z})
\]
   \[= \text{dist}(g_n(y) - j, n\mathbb{Z}) = |g_n(y) - j|.
\]

2. If \( kn + n - y \leq \left\lfloor \frac{n}{2} \right\rfloor \) then \( g_n(y) = kn + n - x \) and
\[
g_n(y - j) = \text{dist}(y - j, n\mathbb{Z}) = \text{dist}(j - y, n\mathbb{Z})
\]
   \[= \text{dist}(kn + n - y + j, n\mathbb{Z}) = g_n(g_n(y) + j)\]
   and
\[
g_n(y + j) = \text{dist}(y + j, n\mathbb{Z}) = \text{dist}(y - j, n\mathbb{Z}) =
\]
   \[= \text{dist}(kn + n - y - j, n\mathbb{Z}) = g_n(g_n(y) - j) = |g_n(y) - j|.
\]

Now we find eigenvectors for the matrix \( a_2 \).

Let \( n \in \mathbb{N} \) and \( 0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \). Let
\[
v_{n,j} = [x_{n,g_n(0)}, x_{n,g_n(j)}, x_{n,g_n(2j)}, \ldots, x_{n,g_n(kj)}, \ldots, x_{n,g_n((n-1)j)}] \in \mathbb{C}^n.
\]
Lemma 6. Let $0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$. Then vector $v_{n,j}$ is an eigenvector of the matrix $a_2$ corresponding to an eigenvalue $x_{n,j}$.

**Proof.** We must show that

\begin{equation}
(*) \quad x_{n,j} \cdot x_{n,g_n(kj)} = x_{n,g_n((k-1)j)} + x_{n,g_n((k+1)j)}
\end{equation}

for $k = 1, 2, \ldots, n - 1$ and

\begin{equation}
(**) \quad x_{n,j} \cdot x_{n,g_n(0)} = x_{n,g_n(j)} + x_{n,g_n((n-1)j)}
\end{equation}

and

\begin{equation}
(***) \quad x_{n,j} \cdot x_{n,g_n((n-1)j)} = x_{n,g_n(0)} + x_{n,g_n((n-2)j)}.
\end{equation}

By Lemma 4 we have

\[ x_{n,j} \cdot x_{n,g_n(kj)} = x_{n,j-g_n(kj)} + x_{n,g_n(j+g_n(kj))}. \]

Hence

\[ x_{n,j} \cdot x_{n,g_n(kj)} = x_{n,g_n((k-1)j)} + x_{n,g_n((k+1)j)}. \]

by Lemma 5 for $y = kj$ and this ends the proof of $(*)$.

Obviously $g_n(0) = 0$ and $g_n(j) = j$. Moreover

\[ g_n((n - 1)j) = \text{dist}(nj - j, n\mathbb{Z}) = \text{dist}(-j, n\mathbb{Z}) = \text{dist}(j, n\mathbb{Z}) = g_n(j). \]

Therefore

\[ x_{n,j} \cdot x_{n,g_n(0)} = x_{n,j} \cdot x_{n,0} = x_{n,j} + x_{n,g_n(j)} = x_{n,g_n(j)} + x_{n,g_n((n-1)j)} \]

and $(**)$ was proved.

We have

\[ x_{n,j} \cdot x_{n,g_n((n-1)j)} = x_{n,j} \cdot x_{n,g_n(j)} = x_{n,j} \cdot x_{n,j} = x_{n,0} + x_{n,g_n(2j)} \]

\[ = x_{n,g_n(0)} + x_{n,g((-2)j)} = x_{n,g_n(0)} + x_{n,g_n((n-2)j)} \]

and $(***)$ was shown. \qed
Notice that if the vector \([y_1, y_2, \ldots, y_n]\) is an eigenvector for the matrix \(a_2\), then the vector \([y_n, y_1, y_2, \ldots, y_{n-1}]\) is also an eigenvector for the matrix \(a_2\).

Let \(n \in \mathbb{N}\) and \(0 \leq j \leq \left\lceil \frac{n}{2} \right\rceil\). Let

\[
\begin{align*}
&u_{n,j} = \\
&[x_{n,g_n((n-1)j)}, x_{n,g_n(0)}, x_{n,g_n(j)}, x_{n,g_n(2j)}, \ldots, x_{n,g_n(kj)}, \ldots, x_{n,g_n((n-2)j)}] \in \mathbb{C}^n.
\end{align*}
\]

Let \(E_{n,j+1} = \text{lin}(v_{n,j}, u_{n,j})\) and \(e_{n,j+1}\) be a matrix of the projection \(\mathbb{C}^n\) onto \(E_{n,j+1}\).

**Lemma 7.**

\[
\dim E_{n,j} = \begin{cases} 
1 & \text{for } j = 1 \text{ or } (j = \frac{n}{2} + 1 \text{ and } 2|n) \\
2 & \text{otherwise.}
\end{cases}
\]

**Proof.** If \(j = 1\) then \(E_{n,1} = \text{lin}(v_{n,0}, u_{n,0}) = \text{lin}([x_{n,0}, \ldots, x_{n,0}], [x_{n,0}, \ldots, x_{n,0}]\), so \(\dim E_{n,1} = 1\).

If \(2|n\) and \(j = \frac{n}{2} + 1\) then \(v_{n,j-1} = [2, -2, 2, \ldots, (-1)^{n+1}]\) (since \(x_{n,\frac{n}{2}} = -2, x_{n,0} = 2\) and \(g_n(\frac{nk}{2}) = 0\) for \(k\) odd and \(g_n(\frac{nk}{2}) = \frac{n}{2}\) for \(k\) even) and \(u_{n,j-1} = (-1)^{n+1}v_{n,j-1}\) hence \(\dim E_{n,j} = 1\).

Otherwise

\[
\det \begin{bmatrix} x_{n,g_n(0)} & x_{n,g_n(j-1)} \\ x_{n,g_n((n-1)(j-1))} & x_{n,g_n(0)} \end{bmatrix} = x_{n,0}^2 - x_{n,j-1}^2 = 4 - x_{n,j-1}^2 \neq 0
\]

hence \(v_{n,j-1}\) and \(u_{n,j-1}\) are linear independent vectors. \(\blacksquare\)

Observe that \(\dim E_{n,1} + \ldots + \dim E_{n,\left\lceil \frac{n}{2} \right\rceil} + 1 = n\) and \(\mathbb{C}^n = E_{n,1} \oplus \ldots \oplus E_{n,\left\lceil \frac{n}{2} \right\rceil} + 1\).

**Lemma 8.** If \(n = 2r + 1\) and \(r > 3\) then

\[
u_n(x) = x^r + x^{r-1} + (1-r)x^{r-2} + \ldots
\]

If \(n = 2r\) and \(r > 2\) then

\[
u_n(x) = x^{r-1} + 0 \cdot x^{r-2} + (2-r)x^{r-3} + \ldots
\]
Proof. $u_5(x) = x^2 + x - 1$ and $u_6(x) = x^2 - 1$. Therefore lemma is true for $n = 5$ and $n = 6$.

If lemma is true for $n = 2r$ and $n = 2r - 1$ then

$$u_{2r+1}(x) = (x + 2)u_{2r} - u_{2r-1}(x)$$

$$= (x + 2)(x^{r-1} + (2 - r)x^{r-2} + \ldots) - (x^{r-1} + x^{r-2} + (1 - (r - 1))x^{r-3} + \ldots)$$

$$= x^r + (2 - 1)x^{r-1} + ((2 - r) - 1)x^{r-2} + \ldots$$

and

$$u_{2r+2}(x) = u_{2r+1}(x) - u_{2r}(x)$$

$$= x^r + x^{r-1} + (1 - r)x^{r-2} + \ldots - (x^{r-1} + (2 - r)x^{r-3} + \ldots)$$

$$= x^r + (1 - r)x^{r-1} + (1 - r - 0)x^{r-2} + \ldots$$

$$= x^r + 0 \cdot x^{r-1} + (1 - r)x^{r-2} + \ldots$$

Hence lemma is true for $n = 2r + 1$ and $n = 2r + 2$.

Lemma 9.

$$x_{n,0}^2 + \ldots + x_{n,\lceil n/2 \rceil}^2 = \begin{cases} n + 2 & \text{for } n \text{ even} \\ n + 4 & \text{for } n \text{ odd} \end{cases}$$

Proof. Consider the following cases:

1. If $n$ is even and $n = 2k + 1$. By Lemma 6 we know that $x_{n,1}, \ldots, x_{n,k}$

are eigenvalues of the matrix $a_2$. Hence they are roots of $w_n$. Obviously

$x_{n,i} \neq 2$ for $i = 1, \ldots, k$, so by Lemma 3 they are roots of $u_n$. Therefore

we have

$$(*) \quad u_n(x) = (x - x_{n,1})(x - x_{n,2}) \cdots (x - x_{n,k}).$$

Using Lemma 8 we obtain $x_{n,1} + \ldots + x_{n,k} = -1$ and $\sum_{1 \leq i < j \leq k} x_{n,i}x_{n,j} = 1 - k$.

Hence

$$x_{n,1}^2 + \ldots + x_{n,k}^2 = (x_{n,1} + \ldots + x_{n,k})^2 - 2 \sum_{1 \leq i < j \leq k} x_{n,i}x_{n,j}$$

$$= 1 - 2(1 - k) = 2k - 1 = n - 2$$

and $x_{n,0}^2 + x_{n,1}^2 + \ldots + x_{n,k}^2 = 4 + n - 2 = n + 2$. 

\[\Box\]
2. Assume that \( n \) is odd and \( n = 2k \). Then

\[
(**) \quad u_n(x) = (x - x_{n,1})(x - x_{n,2}) \ldots (x - x_{n,k-1})
\]

because \( x_{n,1}, \ldots, x_{n,k-1} \) are eigenvalues of the matrix \( a_2 \) by Lemma 6, hence they are roots of \( w_n \) and by Lemma 3 they are also roots of \( u_n \). So by Lemma 8 we have (**).

By Lemma 8 it turns out that \( x_{n,1} + \ldots + x_{n,k-1} = 0 \) and \( \sum_{1 \leq i < j \leq k-1} x_{n,i}x_{n,j} = 2 - k \).

Hence

\[
x_{n,1}^2 + \ldots + x_{n,k-1}^2 = (x_{n,1} + \ldots + x_{n,k-1})^2 - 2 \sum_{1 \leq i < j \leq k-1} x_{n,i}x_{n,j}
\]

\[
= 0 - 2(2 - k) = 2k - 4 = n - 4
\]

and \( x_{n,0}^2 + x_{n,1}^2 + \ldots + x_{n,k-1}^2 + x_{n,k} = 4 + (n - 4) + 4 = n + 4 \).

\[
\]

Lemma 10. Let \( n, k \in \mathbb{N} \) and \( \gcd(n, k) = 1 \). Let \( A = \{0, 1, \ldots, \left[\frac{n}{2}\right]\} \) and \( f : A \to A \) be a function such that \( f(x) = g_n(kx) \). Then \( f \) is a bijection.

\[\textbf{Proof.}\] It is sufficient to show that \( f \) is \( 1 - 1 \). Suppose \( i, j \in A \), \( i < j \) and \( f(i) = f(j) \). Let \( x = \text{dist}(ik, n\mathbb{Z}) = \text{dist}(jk, n\mathbb{Z}) \). There exist \( p, q \in \mathbb{Z} \) such that \( |ik - pn| = |jk - qn| \).

If \( ik - pn = jk - qn \) then \( (i - j)k = (p - q)n \) hence \( n|j - i \) (since \( \gcd(n, k) = 1 \)) but \( j - i \in A \) and we have a contradiction.

If \( ik - pn = -jk + qn \) then \( (i + j)k = (p + q)n \) hence \( n|i + j \) but \( i, j \in A \) so \( 0 < i + j \leq \left[\frac{n}{2}\right] + \left[\frac{n}{2}\right] - 1 < n \) and we obtain a contradiction.

Lemma 11. Let \( n, k, p \in \mathbb{N} \) and \( 0 \leq k \leq \left[\frac{n}{2}\right] \). Then \( |v_{pn,kp}|^2 = p|v_{n,k}|^2 \).

\[\textbf{Proof.}\] Let us note that \( g_{pn}(px) = \text{dist}(px, pn\mathbb{Z}) = p \cdot \text{dist}(x, n\mathbb{Z}) = pg_n(x) \) for any \( x \in \mathbb{Z} \), \( v_{np,kp} = [x_{pn,pg_n((i-1)k)}] \) for \( i = 1, \ldots, pn \) and \( g_n((n + i - 1)k) = g_n((i - 1)k) \).

Consider the following cases:
1. If $2|n$ then $x_{pn,pj} = 2\cos\left(\frac{2\pi j}{pn}\right) = 2\cos\left(\frac{2\pi n}{n}\right) = x_{n,j}$. Hence $v_{np,nk} = [(x_{n, g_n((i-1)k)})_{i=1,...,pn}]$ and $v_{pn,pk} = [v_{n,k}, v_{n,k}, \ldots, v_{n,k}]$ and $|v_{n,k}|^2 = p|v_{n,k}|^2$.

2. If $2 \nmid n$ and $2 \nmid p$ then $x_{pn,pj} = (-1)^{p/2} \cos\left(\frac{2\pi j}{pn}\right) = (-1)^{p/2} \cos\left(\frac{2\pi n}{n}\right) = x_{n,j}, v_{pn,pk} = [v_{n,k}, v_{n,k}, \ldots, v_{n,k}]$ and $|v_{n,k}|^2 = p|v_{n,k}|^2$.

3. If $2 \nmid n$ and $2|p$ then $x_{pn,pj} = 2\cos\left(\frac{2\pi j}{pn}\right) = 2\cos\left(\frac{2\pi n}{n}\right) = 2(2\cos^2\left(\frac{2\pi n}{n}\right) - 1) = 4\cos^2\left(\frac{\pi}{n}\right) - 2 = x_{n,j}^2 - 2$ and by Lemma 4 we have $x_{pn,pj} = x_{n,0} + x_{n,g_n(2j)} - 2 = x_{n,0} + x_{n,g_n(2j)}$. By Lemma 10 $v_{pn,pk} = [v_{n,k}, v_{n,k}, v_{n,k}, \ldots, v_{n,k}]_{p\text{-times}}$ (since $\gcd(2, n) = 1$), where coordinates of $\vec{v}_{n,k}$ arise as a result of the permutation of coordinates of $v_{n,k}$. Hence $|v_{pn,pk}|^2 = p|v_{n,k}|^2$.

\[\begin{align*}
\textbf{Theorem 1.} & \quad \text{Let } n \in \mathbb{N} \text{ and } 0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor. \text{ Then} \\
|v_{n,j}|^2 & = \begin{cases}
4n & \text{for } j = 0 \text{ or } (j = \frac{n}{2} \text{ and } 2|n) \\
2n & \text{otherwise.}
\end{cases}
\end{align*}\]

\textbf{Proof.} Assume that $\gcd(n, j) = 1$.

Let $n = 2r + 1$. According to the fact that $g_n((i-1)k) = g_n((n-i+1)k)$ and by Lemma 10 we have
\[
|v_{n,j}|^2 = |[x_{n,0}, x_{n,1}, \ldots, x_{n,r}, x_{n,1}, \ldots, x_{n,0}]|^2 = 2(x_{n,0}^2 + \ldots + x_{n,r}^2) - x_{n,0}^2 = 2(n + 2) - 4 = 2n
\]

using Lemma 9.

Let $n = 2r$. Then
\[
|v_{n,j}|^2 = |[x_{n,0}, x_{n,1}, \ldots, x_{n,r-1}, x_{n,r}, x_{n,r-1}, \ldots, x_{n,0}]|^2,
\]

by Lemma 10. Hence
\[
|v_{n,j}|^2 = 2(x_{n,0}^2 + \ldots + x_{n,r}^2) - x_{n,0}^2 - x_{n,r}^2 = 2(n + 4) - 4 - 4 = 2n,
\]

by Lemma 9.
Assume that $gcd(n, j) \neq 1$. Let $p = gcd(n, j)$, $n = pm$, $j = pj'$, where $gcd(n', j') = 1$. By Lemma 11 we have $|v_{n,j}|^2 = p|v_{n',j'}|^2$.

One needs to consider the following cases:

1. If $j = 0$ then $v_{n,j} = [2, 2, \ldots, 2]^{n\text{-times}}$ and $|v_{n,j}|^2 = 4n$.

2. If $2\nmid n$ and $j \neq 0$ then $2\nmid n'$ and $|v_{n,j}|^2 = p|v_{n',j'}|^2 = p2n' = 2n$.

3. If $2|n, 2\nmid n'$ and $j \neq 0$ then $2|p$ and $j \neq \frac{n}{2}$ (because if $j = \frac{n}{2}$ then $\frac{p}{2}n' = j = pj' = \frac{p}{2}2j'$ and $n' = 2j'$ but $2\nmid n'$). Hence $|v_{n,j}|^2 = p|v_{n',j'}|^2 = p2n' = 2n$.

4. If $2|n, 2|n'$ and $j' = \frac{n'}{2}$ then $j = pj' = \frac{p}{2}j' = \frac{n}{2}$ and $|v_{n,j}|^2 = p|v_{n',j'}|^2 = p4n' = 4n$.

5. If $2|n, 2|n', j \neq 0$ and $j' \neq \frac{n'}{2}$ then $j' \neq 0, j \neq \frac{n}{2}$ and $|v_{n,j}|^2 = p|v_{n',j'}|^2 = p2n' = 2n$.

\[ \xi_{i,j} = \begin{cases} 
1 & \text{for } i = 1 \text{ or } \left(i = \frac{n}{2} + 1, (2|n) \text{ and } j = 1\right) \\
2 & \text{for } j = 1 \text{ and } i \neq 1 \text{ and } \left(i \text{ if } 2|n \text{ then } i \neq \frac{n}{2} + 1\right) \\
\frac{1}{2}x_{n,gn)((i-1)(j-1)) & \text{for } 2|n \text{ and } i = \frac{n}{2} + 1 \text{ and } j \neq 1 \\
x_{n,gn}((i-1)(j-1)) & \text{otherwise.} 
\end{cases} \]

Let $b = (b_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{C})$ be a matrix. Then let $\tilde{b} = [b_{11}, \ldots, b_{1n}]$. Obviously $\tilde{\cdot}$ is a linear operation.

For $1 \leq i \leq \left[\frac{n}{2}\right]$ let $e_i$ be a matrix of the projection of $\mathbb{C}^n$ onto $E_{n,i}$. We know (see [3]) that $\text{lin}(a_1, \ldots, a_{\left[\frac{n}{2}\right]+1}) = \text{lin}(e_1, \ldots, e_{\left[\frac{n}{2}\right]+1})$.

Let $n \in \mathbb{N}$, $1 \leq i \leq \left[\frac{n}{2}\right]$ and $a_i = \sum_{j=1}^{\left[\frac{n}{2}\right]+1} \xi_{i,j}e_j$.

**Lemma 12.** Let $n \in \mathbb{N}$ and $1 \leq i, j \leq \left[\frac{n}{2}\right] + 1$. Then
Proof. It is obvious that

\[ a_i = \sum_{j=1}^{[n/2]+1} \xi_{ij} \bar{e}_j \quad \text{and} \quad \bar{e}_j = \frac{[1, 0, \ldots, 0] \circ v_{n,j-1}}{|v_{n,j-1}|^2} v_{n,j-1} = \frac{2v_{n,j-1}}{|v_{n,j-1}|^2}, \]

where \( \circ \) means the scalar product of vectors. Using Theorem 1 we have

\[ \bar{e}_j = \begin{cases} 1 & \text{for } j = 1 \text{ or } \left( j = \frac{n}{2} + 1 \text{ and } 2|n \right) \\ \frac{1}{2n} v_{n,j-1} & \text{otherwise.} \end{cases} \]

Hence \( \bar{e}_1, \ldots, \bar{e}_{[n/2]+1} \) are pairwise orthogonal. Therefore \( \xi_{i,j} = \frac{a_i \circ \bar{e}_j}{|\bar{e}_j|^2} \).

Consider the following cases:

1. If \( i = 1 \) and \( j = 1 \) or \( (j = \frac{n}{2} + 1 \text{ and } 2|n) \) then

\[ \xi_{1,j} = \frac{\bar{a}_1 \circ \bar{e}_j}{|\bar{e}_j|^2} = \frac{\frac{1}{2n} v_{n,j-1}}{\frac{1}{4n^2} |v_{n,j-1}|} = \frac{1}{n} = 1. \]

2. If \( i = 1 \) and \( j \neq 1 \) and \( (j \neq \frac{n}{2} + 1 \text{ if } 2|n) \) then

\[ \xi_{1,j} = \frac{\bar{a}_1 \circ \bar{e}_j}{|\bar{e}_j|^2} = \frac{\frac{1}{n} v_{n,j-1}}{\frac{1}{4n^2} |v_{n,j-1}|} = \frac{2}{n^2} = 1. \]

3. If \( i = \frac{n}{2} + 1, 2|n \) and \( j = 1 \) then

\[ \xi_{i,1} = \frac{\bar{a}_i \circ \bar{e}_1}{|\bar{e}_1|^2} = \frac{\frac{1}{n}}{\frac{1}{4n^2}} = 1. \]

4. If \( j = 1 \) and \( i \neq 1 \) and \( (i \neq \frac{n}{2} + 1 \text{ if } 2|n) \) then

\[ \xi_{i,1} = \frac{\bar{a}_i \circ \bar{e}_1}{|\bar{e}_1|^2} = \frac{\frac{2}{n^2}}{\frac{4n^2}{4n^2}} = 2. \]
5. If $2 \mid n$ and $i = \frac{n}{2} + 1$, $j \neq 1$ and $j \neq \frac{n}{2} + 1$ then
\[
\xi_{i,j} = \frac{\bar{a}_i \circ \bar{e}_j}{|\bar{e}_j|^2} = \frac{1}{n} x_{n, g_n}\left(\frac{n}{2}(j-1)\right) = \frac{1}{2} x_{n, g_n}\left(\frac{n}{2}(j-1)\right).
\]

6. If $2 \mid n$ and $i = \frac{n}{2} + 1$ and $j = \frac{n}{2} + 1$ then
\[
\xi_{i,j} = \frac{\bar{a}_i \circ \bar{e}_j}{|\bar{e}_j|^2} = \frac{1}{2^n} x_{n, g_n}\left(\frac{n}{2} \frac{n}{2}\right) = \frac{1}{2} x_{n, g_n}\left(\frac{n}{2} \frac{n}{2}\right).
\]

7. If $2 \mid n$ and $j = \frac{n}{2} + 1$, $i \neq 1$ and $i \neq \frac{n}{2} + 1$ then $\bar{a}_i = [b_1, \ldots, b_n]$, where
\[
b_j = \begin{cases} 
1 & \text{for } j = i \text{ or } j = n - i + 2 \\
0 & \text{for } j \neq i \text{ and } j \neq n - i + 2.
\end{cases}
\]

Moreover $\nu_{\frac{n}{2}} = [2, -2, \ldots, 2(-1)^{n+1}]$. Hence
\[
\xi_{i,j} = \frac{\frac{1}{2^n}(2(-1)^{i+1} + 2(-1)^{n-i+1})}{\frac{4}{3n^2} 4n} = 2(-1)^{i+1} = x_{n, g_n}(\frac{n}{2} \frac{n}{2}).
\]

8. If $i \neq 1$, $j \neq 1$, $(i \neq \frac{n}{2} + 1$ and $j \neq \frac{n}{2} + 1$ if $2 \mid n$) then
\[
\xi_{i,j} = \frac{1}{n} \left( x_{n, g_n}(i-1)(j-1) + x_{n, g_n}(n-i+1)(j-1) \right) \left( \frac{1}{n} 2n \right)
\]
\[
= \frac{n}{2} x_{n, g_n}(i-1)(j-1) \left( \frac{1}{n^2} 2n \right) = x_{n, g_n}(i-1)(j-1).
\]

Let $f_i = dim \mathbb{C} E_{n,i}$, $n_j = \frac{|C_j|}{n}$ and $\varphi_{i,j} = \sqrt{f_i} \xi_{i,j} n_j^{-1}$ for $i, j \in \{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}$. Then $(\varphi_{i,j})_{1 \leq i,j \leq \left\lfloor \frac{n}{2} \right\rfloor}$ is the character table of the quasigroup $(\mathbb{Z}_n, -n)$.

The next Theorem gives the description of the character table of the quasigroup $(\mathbb{Z}_n, -n)$. 
Theorem 2. Let \( n \in \mathbb{N} \) and \( 1 \leq i, j \leq \left[ \frac{n}{2} \right] + 1 \). Then

\[
\varphi_{i,j} = \begin{cases} 
1 & \text{for } i = 1 \text{ or } (i = \frac{n}{2} + 1, (2|n) \text{ and } j = 1) \\
\sqrt{2} & \text{for } j = 1 \text{ and } i \neq 1 \text{ and } \left( \text{if } 2|n \text{ then } i \neq \frac{n}{2} + 1 \right) \\
(-1)^{j-1} & \text{for } 2|n \text{ and } i = \frac{n}{2} + 1 \text{ and } j \neq 1 \\
\frac{\sqrt{2}}{2} x_{n,g_n((i-1)(j-1))} & \text{otherwise.}
\end{cases}
\]

Hence for \( n \) even we obtain

\[
\begin{array}{c|c|c}
  & j = 1 & j \neq 1 \\
i = 1 & \varphi_{i,j} = 1 & \varphi_{i,j} = 1 \\
i \neq 1 & \varphi_{i,j} = \sqrt{2} x_{n,g_n((i-1)(j-1))} & \varphi_{i,j} = \sqrt{2} x_{n,g_n((i-1)(j-1))}
\end{array}
\]

and for \( n \) odd we have

\[
\begin{array}{c|c|c|c}
  & j = 1 & j \neq 1, j \neq \frac{n}{2} + 1 & j = \frac{n}{2} + 1 \\
i = 1 & \varphi_{i,j} = 1 & \varphi_{i,j} = 1 & \varphi_{i,j} = 1 \\
i \neq 1, i \neq \frac{n}{2} + 1 & \varphi_{i,j} = \sqrt{2} x_{n,g_n((i-1)(j-1))} & \varphi_{i,j} = \sqrt{2} x_{n,g_n((i-1)(j-1))} & \varphi_{i,j} = \sqrt{2} x_{n,g_n((i-1)(j-1))} \\
i = \frac{n}{2} + 1 & \varphi_{i,j} = 1 & \varphi_{i,j} = (-1)^{j-1} & \varphi_{i,j} = (-1)^{\frac{n}{2}}
\end{array}
\]

Proof. We must consider the following cases (we use Lemma 7 to calculate \( f_i \)):

1. If \( i = 1 \) and \( (j = 1 \text{ or } (j = \frac{n}{2} + 1 \text{ and } 2|n)) \) then

\[
\varphi_{i,j} = \sqrt{1} \xi_{j,i} \frac{n}{n} = 1.
\]
2. If \( i = 1, j \neq 1 \) and \((2|n \text { then } j \neq \frac{n}{2} + 1)\) then

\[
\varphi_{i,j} = \sqrt{\frac{1}{2}} \xi_{j,i} \frac{n}{2n} = 1.
\]

3. If \( 2|n, i = \frac{n}{2} + 1 \) and \( j = 1 \) then

\[
\varphi_{i,j} = \sqrt{\frac{1}{2}} \xi_{j,i} \frac{n}{n} = 1.
\]

4. If \( 2|n, i = \frac{n}{2} + 1 \) and \( j = \frac{n}{2} + 1 \) then

\[
\varphi_{i,j} = \sqrt{\frac{1}{2}} \xi_{j,i} \frac{n}{2n} = \frac{1}{2} x_{n, g_n} \left( \frac{n}{2} \right) = (-1)^{\frac{n}{2}} = (-1)^{j-1}.
\]

5. If \( 2|n, i = \frac{n}{2} + 1 \) and \( j \neq 1 \) and \( j \neq \frac{n}{2} + 1 \) then

\[
\varphi_{i,j} = \sqrt{\frac{1}{2}} \xi_{j,i} \frac{n}{2n} = \frac{1}{2} x_{n, g_n} \left( j-1 \frac{n}{2} \right) = (-1)^{j-1}.
\]

6. If \( i \neq 1 \) and \( j = 1 \) and \((2|n \text { then } i \neq \frac{n}{2} + 1)\) then

\[
\varphi_{i,j} = \sqrt{2} \xi_{j,i} \frac{n}{n} = \sqrt{2}.
\]

7. If \( i \neq 1 \) and \( i \neq \frac{n}{2} + 1 \) and \( 2|n \text { and } j = \frac{n}{2} + 1 \) then

\[
\varphi_{i,j} = \sqrt{2} \xi_{j,i} \frac{n}{n} = \sqrt{2} \frac{1}{2} x_{n, g_n} \left( \frac{1}{2} (i-1) \right) = \sqrt{2} (-1)^{\frac{n}{2}} = \sqrt{2} \frac{1}{2} x_{n, g_n} \left( (i-1)(j-1) \right).
\]

8. If \( i \neq 1 \) and \((2|n \text { then } i \neq \frac{n}{2} + 1)\) and \( j \neq 1 \) and \((2|n \text { then } j \neq \frac{n}{2} + 1)\) then

\[
\varphi_{i,j} = \sqrt{2} \xi_{j,i} \frac{n}{2n} = \frac{\sqrt{2}}{2} x_{n, g_n} \left( (i-1)(j-1) \right).
\]
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