ON FUZZY IDEALS OF PSEUDO MV-ALGEBRAS

Grzegorz Dymek

Institute of Mathematics and Physics
University of Podlasie
3 Maja 54, 08–110 Siedlce, Poland
e-mail: gdymek@o2.pl

Abstract

Fuzzy ideals of pseudo MV-algebras are investigated. The homomorphic properties of fuzzy prime ideals are given. A one-to-one correspondence between the set of maximal ideals and the set of fuzzy maximal ideals \( \mu \) satisfying \( \mu(0) = 1 \) and \( \mu(1) = 0 \) is obtained.

Keywords: pseudo MV-algebra, fuzzy (prime, maximal) ideal.

2000 Mathematics Subject Classification: 06D35.

1. Introduction

The study of pseudo MV-algebras was initiated by G. Georgescu and A. Iorgulescu in [5] and [6], and independently by J. Rachunek in [9] (there they are called generalized MV-algebras or, in short, GMV-algebras) as a non-commutative generalization of MV-algebras which were introduced by C.C. Chang in [1]. The concept of a fuzzy set was introduced by L.A. Zadeh in [10]. Since then these ideas have been applied to other algebraic structures such as semigroups, groups, rings, ideals, modules, vector spaces and topologies. In [8] Y.B. Jun and A. Walendziak applied the concept of a fuzzy set to pseudo MV-algebras. They introduced the notions of a fuzzy ideal and a fuzzy implicative ideal in a pseudo MV-algebra, gave characterizations of them and provided conditions for a fuzzy set to be a fuzzy ideal and a fuzzy implicative ideal. Recently, the author in [3] and [4] defined, investigated and characterized fuzzy prime and fuzzy maximal ideals of pseudo MV-algebras.
In the paper we conduct further investigations of these ideals in Section 3. We provide the homomorphic properties of fuzzy prime ideals. A one-to-one correspondence between the set of maximal ideals of a pseudo $MV$-algebra $A$ and the set of fuzzy maximal ideals $\mu$ of $A$ such that $\mu(0) = 1$ and $\mu(1) = 0$ is established. For the convenience of the reader, in Section 2 we give the necessary material needed in sequel, thus making our exposition self-contained.

2. Preliminaries

Let $A = (A, \oplus, -, \sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. For any $x, y \in A$, set $x \cdot y = (y^- \oplus x^-)\sim$. We consider that the operation $\cdot$ has priority to the operation $\oplus$, i.e., we will write $x \oplus y \cdot z$ instead of $x \oplus (y \cdot z)$. The algebra $A$ is called a pseudo $MV$-algebra if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(A2) $x \oplus 0 = 0 \oplus x = x$,

(A3) $x \oplus 1 = 1 \oplus x = 1$,

(A4) $1^- = 0, 1^- = 0$,

(A5) $(x^- \oplus y^-)\sim = (x^- \oplus y^-)\sim$,

(A6) $x \oplus x^- \cdot y = y \oplus y^- \cdot x = x \cdot y^- \oplus y = y \cdot x^- \oplus x$,

(A7) $x \cdot (x^- \oplus y) = (x \oplus y^-) \cdot y$,

(A8) $(x^-)\sim = x$.

If the addition $\oplus$ is commutative, then both unary operations $-$ and $\sim$ coincide and $A$ is an $MV$-algebra.

Throughout this paper $A$ will denote a pseudo $MV$-algebra. For any $x \in A$ and $n = 0, 1, 2, \ldots$ we put

$$0x = 0 \text{ and } (n+1)x = nx \oplus x,$$

$$x^0 = 1 \text{ and } x^{n+1} = x^n \cdot x.$$
Proposition 2.1 (Georgescu and Iorgulescu [6]). The following properties hold for any \( x \in A \):

\[
\begin{align*}
(a) \quad (x^\sim)^- &= x, \\
(b) \quad x \oplus x^\sim &= 1, \quad x^- \oplus x = 1, \\
(c) \quad x \cdot x^- &= 0, \quad x^\sim \cdot x = 0.
\end{align*}
\]

We define \( x \leq y \) iff \( x^- \oplus y = 1 \).

Proposition 2.2 (Georgescu and Iorgulescu [6]). The following properties hold for any \( a, x, y \in A \):

\[
\begin{align*}
(a) \quad \text{if } x \leq y, \text{ then } a \oplus x &\leq a \oplus y, \\
(b) \quad \text{if } x \leq y, \text{ then } x \oplus a &\leq y \oplus a.
\end{align*}
\]

As it is shown in [6], \( (A, \leq) \) is a lattice in which the join \( x \lor y \) and the meet \( x \land y \) of any two elements \( x \) and \( y \) are given by:

\[
\begin{align*}
x \lor y &= x \oplus x^\sim \cdot y = x \cdot y^- \oplus y, \\
x \land y &= x \cdot (x^- \oplus y) = (x \oplus y^-) \cdot y.
\end{align*}
\]

Definition 2.3. A subset \( I \) of \( A \) is called an ideal of \( A \) if it satisfies:

\[
\begin{align*}
(I1) & \quad 0 \in I, \\
(I2) & \quad \text{if } x, y \in I, \text{ then } x \oplus y \in I, \\
(I3) & \quad \text{if } x \in I, \text{ } y \in A \text{ and } y \leq x, \text{ then } y \in I.
\end{align*}
\]

Denote by \( \mathcal{I}(A) \) the set of all ideals of \( A \).

Remark 2.4. Let \( I \in \mathcal{I}(A) \). If \( x, y \in I \), then \( x \cdot y, x \land y, x \lor y \in I \).

Definition 2.5. Let \( I \) be a proper ideal of \( A \) (i.e., \( I \neq A \)). Then
(a) \( I \) is called prime if, for all \( I_1, I_2 \in \mathcal{I}(A) \), \( I = I_1 \cap I_2 \) implies \( I = I_1 \) or \( I = I_2 \).

(b) \( I \) is called maximal iff whenever \( J \) is an ideal such that \( I \subseteq J \subseteq A \), then either \( J = I \) or \( J = A \).

Denote by \( \mathcal{M}(A) \) the set of all maximal ideals of \( A \).

**Definition 2.6.** The order of an element \( x \in A \) is the least \( n \) such that \( nx = 1 \) if such \( n \) exists, and \( \text{ord}(x) = \infty \) otherwise.

**Remark 2.7.** It is easy to see that for any \( x \in A \), \( \text{ord}(x^-) = \text{ord}(x^\sim) \).

**Theorem 2.8.** Let \( x \in A \). Then \( \text{ord}(x) = \infty \) if and only if \( x \in I \) for some proper ideal \( I \) of \( A \).

**Proof.** Let \( x \in A \). If \( x \) belongs to a proper ideal of \( A \), then clearly \( \text{ord}(x) = \infty \). Now, assume that \( \text{ord}(x) = \infty \). Let \( I \) be the set of all elements \( y \) such that \( y \leq nx \) for some \( n \in \mathbb{N} \). Then \( x \in I \) and \( I \) is a proper ideal of \( A \).

**Definition 2.9.** Let \( A \) and \( B \) be pseudo MV-algebras. A function \( f : A \to B \) is a homomorphism if and only if it satisfies, for each \( x, y \in A \), the following conditions:

\begin{align*}
\text{(H1)} & \quad f(0) = 0, \\
\text{(H2)} & \quad f(x \oplus y) = f(x) \oplus f(y), \\
\text{(H3)} & \quad f(x^-) = (f(x))^-, \\
\text{(H4)} & \quad f(x^\sim) = (f(x))^\sim.
\end{align*}

**Remark 2.10.** We also have for all \( x, y \in A \):

(a) \( f(1) = 1 \),

(b) \( f(x \cdot y) = f(x) \cdot f(y) \),

(c) \( f(x \lor y) = f(x) \lor f(y) \),

(d) \( f(x \land y) = f(x) \land f(y) \).
We now review some fuzzy logic concepts. Let $\Gamma$ be a subset of the interval $[0, 1]$ of real numbers. We define $\bigwedge \Gamma = \inf \Gamma$ and $\bigvee \Gamma = \sup \Gamma$. Obviously, if $\Gamma = \{\alpha, \beta\}$, then $\alpha \wedge \beta = \min \{\alpha, \beta\}$ and $\alpha \vee \beta = \max \{\alpha, \beta\}$. Recall that a fuzzy set in $A$ is a function $\mu : A \rightarrow [0, 1]$. For any fuzzy sets $\mu$ and $\nu$ in $A$, we define

$$\mu \leq \nu \iff \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

**Definition 2.11.** Let $A$ and $B$ be any two sets, $\mu$ be any fuzzy set in $A$ and $f : A \rightarrow B$ be any function. The fuzzy set $\nu$ in $B$ defined by

$$\nu(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\
0 & \text{otherwise}
\end{cases}$$

for all $y \in B$, is called the image of $\mu$ under $f$ and is denoted by $f(\mu)$.

**Definition 2.12.** Let $A$ and $B$ be any two sets, $f : A \rightarrow B$ be any function and $\nu$ be any fuzzy set in $f(A)$. The fuzzy set $\mu$ in $A$ defined by

$$\mu(x) = \nu(f(x)) \text{ for all } x \in A$$

is called the preimage of $\nu$ under $f$ and is denoted by $f^{-1}(\nu)$.

### 3. Fuzzy ideals

In this section we investigate fuzzy prime ideals and fuzzy maximal ideals of a pseudo $MV$-algebra. First, we recall from [8] the definition and some facts concerning fuzzy ideals.

**Definition 3.1.** A fuzzy set $\mu$ in a pseudo $MV$-algebra $A$ is called a fuzzy ideal of $A$ if it satisfies for all $x, y \in A$:

1. $\mu(x \oplus y) \geq \mu(x) \wedge \mu(y)$,
2. if $y \leq x$, then $\mu(y) \geq \mu(x)$.
It is easily seen that (d2) implies

(d3) \( \mu(0) \geq \mu(x) \) for all \( x \in A \).

Denote by \( \mathcal{FI}(A) \) the set of all fuzzy ideals of \( A \).

**Example 3.2.** Let \( A = \{(1, y) \in \mathbb{R}^2 : y \geq 0\} \cup \{(2, y) \in \mathbb{R}^2 : y \leq 0\} \), \( 0 = (1, 0), \ 1 = (2, 0) \). For any \((a, b), (c, d) \in A\), we define operations \(\oplus, \neg, \sim\) as follows:

\[
(a, b) \oplus (c, d) = \begin{cases} 
(1, b + d) & \text{if } a = c = 1, \\
(2, ad + b) & \text{if } ac = 2 \text{ and } ad + b \leq 0, \\
(2, 0) & \text{in other cases,}
\end{cases}
\]

\[
(a, b)^- = \left( \frac{2}{a}, -\frac{2b}{a} \right),
\]

\[
(a, b)^\sim = \left( \frac{2}{a}, -\frac{b}{a} \right).
\]

Then \( A = (A, \oplus, \neg, \sim, 0, 1) \) is a pseudo MV-algebra (see [2]). Let \( A_1 = \{(1, y) \in \mathbb{R}^2 : y > 0\} \) and \( A_2 = \{(2, y) \in \mathbb{R}^2 : y < 0\} \) and let \( 0 \leq \alpha_3 < \alpha_2 < \alpha_1 \leq 1 \). We define a fuzzy set \( \mu \) in \( A \) as follows:

\[
\mu(x) = \begin{cases} 
\alpha_1 & \text{if } x = 0, \\
\alpha_2 & \text{if } x \in A_1, \\
\alpha_3 & \text{if } x \in A_2 \cup \{1\}.
\end{cases}
\]

It is easily checked that \( \mu \) satisfies (d1) and (d2). Thus \( \mu \in \mathcal{FI}(A) \).

**Proposition 3.3** (Jun and Walendziak [8]). Every fuzzy ideal \( \mu \) of \( A \) satisfies the following two inequalities:

\[
\begin{align*}
(1) \quad \mu(y) & \geq \mu(x) \land \mu(y \cdot x^-) \quad \text{for all } x, y \in A, \\
(2) \quad \mu(y) & \geq \mu(x) \land \mu(x^\sim \cdot y) \quad \text{for all } x, y \in A.
\end{align*}
\]
Proposition 3.4 (Jun and Walendziak [8]). For a fuzzy set \( \mu \) in \( A \), the following are equivalent:

(a) \( \mu \in \mathcal{FI}(A) \),

(b) \( \mu \) satisfies the conditions (d3) and (1),

(c) \( \mu \) satisfies the conditions (d3) and (2).

Now, we consider two special fuzzy sets in \( A \). Let \( I \) be a subset of \( A \). Define a fuzzy set \( \mu_I \) in \( A \) by

\[
\mu_I(x) = \begin{cases} 
\alpha & \text{if } x \in I, \\
\beta & \text{otherwise},
\end{cases}
\]

where \( \alpha, \beta \in [0,1] \) with \( \alpha > \beta \). The fuzzy set \( \mu_I \) is a generalization of a fuzzy set \( \chi_I \) which is the characteristic function of \( I \):

\[
\chi_I(x) = \begin{cases} 
1 & \text{if } x \in I, \\
0 & \text{otherwise}.
\end{cases}
\]

We have simple proposition.

Proposition 3.5. \( I \in \mathcal{I}(A) \) iff \( \mu_I \in \mathcal{FI}(A) \).

Corollary 3.6. \( I \in \mathcal{I}(A) \) iff \( \chi_I \in \mathcal{FI}(A) \).

For an arbitrary fuzzy set \( \mu \) in \( A \), consider the set \( A_\mu = \{ x \in A : \mu(x) = \mu(0) \} \). We have the following simple proposition.

Proposition 3.7. If \( \mu \in \mathcal{FI}(A) \), then \( A_\mu \in \mathcal{I}(A) \).

The following example shows that the converse of Proposition 3.7 does not hold.

Example 3.8. Let \( A \) be as in Example 3.2. Define a fuzzy set \( \mu \) in \( A \) by

\[
\mu(x) = \begin{cases} 
\frac{1}{2} & \text{if } x = 0, \\
\frac{2}{3} & \text{if } x \neq 0.
\end{cases}
\]

Then \( A_\mu = \{0\} \in \mathcal{I}(A) \) but \( \mu \notin \mathcal{FI}(A) \).
Since $A_{\mu_I} = I$, we have a simple proposition.

**Proposition 3.9.** $\mu_I \in \mathcal{FI}(A)$ iff $A_{\mu_I} \in \mathcal{I}(A)$.

**Proposition 3.10.** Let $\mu, \nu \in \mathcal{FI}(A)$. If $\mu \leq \nu$ and $\mu(0) = \nu(0)$, then $A_\mu \subseteq A_\nu$.

**Proof.** Let $x \in A_\mu$. Then $\mu(x) = \mu(0) = \nu(0)$ and since $\mu(x) \leq \nu(x)$, we have $\nu(x) = \nu(0)$. Hence, $x \in A_\nu$.

**Theorem 3.11.** Let $x \in A$. Then $\text{ord}(x) = \infty$ if and only if $\mu(x) = \mu(0)$ for some non-constant fuzzy ideal $\mu$ of $A$.

**Proof.** Let $x \in A$. Suppose $\text{ord}(x) = \infty$. Then, by Theorem 2.8, $x \in I$ for some proper ideal $I$ of $A$. Thus $\chi_I(x) = 1 = \chi_I(0)$ for the non-constant fuzzy ideal $\chi_I$ of $A$.

Conversely, assume that $\mu(x) = \mu(0)$ for some non-constant fuzzy ideal $\mu$ of $A$. Then $x \in A_\mu$ and $A_\mu$ is a proper ideal of $A$. Hence, by Theorem 2.8, $\text{ord}(x) = \infty$.

**Theorem 3.12.** Let $\mu \in \mathcal{FI}(A)$. Then a subset $P(\mu) = \{x \in A : \mu(x) > 0\}$ of $A$ is an ideal when it is non-empty.

**Proof.** Assume that $\mu$ is a fuzzy ideal of $A$ such that $P(\mu) \neq \emptyset$. Obviously, $0 \in P(\mu)$. Let $x, y \in A$ be such that $x, y \in P(\mu)$. Then $\mu(x) > 0$ and $\mu(y) > 0$. It follows from (d1) that $\mu(x \oplus y) \geq \mu(x) \wedge \mu(y) > 0$ so that $x \oplus y \in P(\mu)$. Now, let $x, y \in A$ be such that $x \in P(\mu)$ and $y \leq x$. Then, by (d2), we have $\mu(y) \geq \mu(x)$, and since $\mu(x) > 0$, we obtain $\mu(y) > 0$. So, $y \in P(\mu)$. Thus, $P(\mu)$ is the ideal of $A$.

**Proposition 3.13** (Dymek [3]). Let $f : A \rightarrow B$ be a homomorphism, $\mu \in \mathcal{FI}(A)$ and $\nu \in \mathcal{FI}(B)$. Then:

(a) if $\mu$ is constant on $\ker f$, then $f^{-1}(f(\mu)) = \mu$,

(b) if $f$ is surjective, then $f(f^{-1}(\nu)) = \nu$.

**Proposition 3.14** (Dymek [3]). Let $f : A \rightarrow B$ be a surjective homomorphism and $\nu \in \mathcal{FI}(B)$. Then $f^{-1}(\nu) \in \mathcal{FI}(A)$. 
Proposition 3.15 (Dymek [3]). Let $f : A \rightarrow B$ be a surjective homomorphism and $\mu \in \mathcal{J}(A)$ be such that $A_\mu \supseteq \text{Ker} f$. Then $f(\mu) \in \mathcal{J}(B)$.

Now, we establish the analogous homomorphic properties of fuzzy prime ideals. First, we recall from [4] the definition and some characterizations of a fuzzy prime ideal.

Definition 3.16. A fuzzy ideal $\mu$ of $A$ is said to be fuzzy prime if it is non-constant and satisfies:

$$\mu(x \land y) = \mu(x) \lor \mu(y)$$

for all $x, y \in A$.

Proposition 3.17 (Dymek [4]). Let $\mu$ be a non-constant fuzzy ideal of $A$. Then the following are equivalent:

(a) $\mu$ is a fuzzy prime ideal of $A$,
(b) for all $x, y \in A$, if $\mu(x \land y) = \mu(0)$, then $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$,
(c) for all $x, y \in A$, $\mu(x \cdot y^-) = \mu(0)$ or $\mu(y \cdot x^-) = \mu(0)$,
(d) for all $x, y \in A$, $\mu(x^\sim \cdot y) = \mu(0)$ or $\mu(y^\sim \cdot x) = \mu(0)$.

The following two theorems give the homomorphic properties of fuzzy prime ideals and they are a supplement of the Section 4 of [4].

Theorem 3.18. Let $f : A \rightarrow B$ be a surjective homomorphism and $\nu$ be a fuzzy prime ideal of $B$. Then $f^{-1}(\nu)$ is a fuzzy prime ideal of $A$.

Proof. From Proposition 3.14 we know that $f^{-1}(\nu) \in \mathcal{J}(A)$. Obviously, $f^{-1}(\nu)$ is non-constant. Let $x, y \in A$ be such that $(f^{-1}(\nu))(x \land y) = (f^{-1}(\nu))(0)$. Then $\nu(f(x) \land f(y)) = \nu(f(0)) = \nu(0)$. So, by Proposition 3.17, $\nu(f(x)) = \nu(f(0))$ or $\nu(f(y)) = \nu(f(0))$, i.e., $(f^{-1}(\nu))(x) = (f^{-1}(\nu))(0)$ or $(f^{-1}(\nu))(y) = (f^{-1}(\nu))(0)$. Therefore, from Proposition 3.17 it follows that $f^{-1}(\nu)$ is a fuzzy prime ideal of $A$.

Theorem 3.19. Let $f : A \rightarrow B$ be a surjective homomorphism and $\mu$ a fuzzy prime ideal of $A$ such that $A_\mu \supseteq \text{Ker} f$. Then $f(\mu)$ is a fuzzy prime ideal of $B$ when it is non-constant.
Proof. From Proposition 3.15 we know that $f(\mu) \in \mathcal{F}(A)$. Assume that $f(\mu)$ is non-constant. Let $x_B, y_B \in B$ be such that $(f(\mu))(x_B \wedge y_B) = (f(\mu))(0)$. Since $f$ is surjective, there exist $x_A, y_A \in A$ such that $f(x_A) = x_B$ and $f(y_A) = y_B$. Since $A_\mu \supseteq \text{Ker} f$, $\mu$ is constant on $\text{Ker} f$. Hence, by Proposition 3.13(a), we have

$$\mu(0) = (f(\mu))(0) = (f(\mu))(x_B \wedge y_B) = (f(\mu))(f(x_A \wedge y_A))$$

$$= (f^{-1}(f(\mu)))(x_A \wedge y_A) = \mu(x_A \wedge y_A).$$

Since $\mu$ is fuzzy prime, from Proposition 3.17 we conclude that $\mu(x_A) = \mu(0)$ or $\mu(y_A) = \mu(0)$. Thus

$$(f(\mu))(0) = \mu(0) = \mu(x_A) = (f^{-1}(f(\mu)))(x_A)$$

$$= (f(\mu))(f(x_A)) = (f(\mu))(x_B) \text{ or}$$

$$(f(\mu))(0) = \mu(0) = \mu(y_A) = (f^{-1}(f(\mu)))(y_A)$$

$$= (f(\mu))(f(y_A)) = (f(\mu))(y_B).$$

Therefore, from Proposition 3.17 it follows that $f(\mu)$ is a fuzzy prime ideal of $A$. 

Now, we investigate fuzzy maximal ideals of a pseudo MV-algebra. The investigations are a continuation of the Section 4 of [3].

Definition 3.20. A fuzzy ideal $\mu$ of $A$ is called fuzzy maximal iff $A_\mu$ is a maximal ideal of $A$.

Denote by $\mathcal{FM}(A)$ the set of all fuzzy maximal ideals of $A$.

Proposition 3.21 (Dymek [3]). Let $I \in \mathcal{I}(A)$. Then $I \in \mathcal{M}(A)$ if and only if $\mu_I \in \mathcal{FM}(A)$.

Corollary 3.22. Let $I \in \mathcal{I}(A)$. Then $I \in \mathcal{M}(A)$ if and only if $\chi_I \in \mathcal{FM}(A)$.

Proposition 3.23 (Dymek [3]). If $\mu \in \mathcal{FM}(A)$, then $\mu$ has exactly two values.
Now, denote by $\mathcal{FM}_0(A)$ the set of all fuzzy maximal ideals $\mu$ of $A$ such that $\mu(0) = 1$ and $\mu(1) = 0$. Obviously, $\mathcal{FM}_0(A) \subseteq \mathcal{FM}(A)$. From Proposition 3.23 we immediately have the following theorem.

**Theorem 3.24.** If $\mu \in \mathcal{FM}_0(A)$, then $\text{Im}\mu = \{0, 1\}$.

**Theorem 3.25.** If $\mu \in \mathcal{FM}_0(A)$, then $\mu = \chi_{A_\mu}$.

**Proof.** Let $x \in A$. Since

$$\chi_{A_\mu}(x) = \begin{cases} 1 & \text{if } \mu(x) = 1, \\ 0 & \text{if } \mu(x) = 0, \end{cases}$$

we have the result.

**Theorem 3.26.** If $\mu \in \mathcal{FM}_0(A)$, then $A_\mu = P(\mu)$.

**Proof.** It is straightforward.

**Theorem 3.27.** Let $\mu \in \mathcal{FM}_0(A)$. If there exists a fuzzy ideal $\nu$ of $A$ such that $\nu(0) = 1, \nu(1) = 0$ and $\mu \leq \nu$, then $\nu \in \mathcal{FM}_0(A)$ and $\mu = \nu = \chi_{A_\mu} = \chi_{A_\nu}$.

**Proof.** From Proposition 3.10 we know that $A_\mu \subseteq A_\nu$. Since $A_\mu$ is maximal, it follows that $A_\mu = A_\nu$ because $A_\nu \neq A$. Thus $A_\nu$ is also maximal. Hence $\nu$ is fuzzy maximal, and so $\nu \in \mathcal{FM}_0(A)$. Since $\mu, \nu \in \mathcal{FM}_0(A)$, by Theorem 3.25, $\mu = \chi_{A_\mu}$ and $\nu = \chi_{A_\nu}$. Thus $\mu = \chi_{A_\mu} = \chi_{A_\nu} = \nu$.

**Theorem 3.28.** Let $\mu \in \mathcal{FM}(A)$ and define a fuzzy set $\hat{\mu}$ in $A$ by

$$\hat{\mu}(x) = \frac{\mu(x) - \mu(1)}{\mu(0) - \mu(1)}$$

for all $x \in A$. Then $\hat{\mu} \in \mathcal{FM}_0(A)$.

**Proof.** Since $\mu(0) \geq \mu(x)$ for all $x \in A$ and $\mu(0) \neq \mu(1)$, $\hat{\mu}$ is well-defined. Clearly, $\hat{\mu}(1) = 0$ and $\hat{\mu}(0) = 1 \geq \hat{\mu}(x)$ for all $x \in A$. Thus $\hat{\mu}$ satisfies (d3).
Let \( x, y \in A \). Then
\[
\hat{\mu}(x) \land \hat{\mu}(y \cdot x^-) = \frac{\mu(x) - \mu(1)}{\mu(0) - \mu(1)} \lor \frac{\mu(y \cdot x^-) - \mu(1)}{\mu(0) - \mu(1)}
\]
\[
= \frac{1}{\mu(0) - \mu(1)} \left[ (\mu(x) - \mu(1)) \lor (\mu(y \cdot x^-) - \mu(1)) \right]
\]
\[
= \frac{1}{\mu(0) - \mu(1)} \left[ (\mu(x) \lor \mu(y \cdot x^-)) - \mu(1) \right]
\]
\[
\leq \frac{1}{\mu(0) - \mu(1)} [\mu(y) - \mu(1)] = \frac{\mu(y) - \mu(1)}{\mu(0) - \mu(1)} = \hat{\mu}(y).
\]

Thus \( \hat{\mu} \) satisfies (1). Therefore, \( \hat{\mu} \) is the fuzzy ideal of \( A \) satisfying \( \hat{\mu}(0) = 1 \) and \( \hat{\mu}(1) = 0 \). Moreover, it is easily seen, that \( A_{\hat{\mu}} = A_{\mu} \). Hence, \( \hat{\mu} \in FM_0(A) \).

Corollary 3.29. If \( \mu \in FM_0(A) \), then \( \mu = \hat{\mu} \).

Now, we show a one-to-one correspondence between the sets \( M(A) \) and \( FM_0(A) \).

Theorem 3.30. Let \( A \) be a pseudo MV-algebra. Then functions \( \varphi : M(A) \to FM_0(A) \) defined by \( \varphi(M) = \chi_M \) for all \( M \in M(A) \) and \( \psi : FM_0(A) \to M(A) \) defined by \( \psi(\mu) = A_\mu \) for all \( \mu \in FM_0(A) \) are inverses of each other.

Proof. Let \( M \in M(A) \). Then \( \psi \varphi(M) = \psi(\chi_M) = A_{\chi_M} = M \). Now, let \( \mu \in FM_0(A) \). Then we also have \( \varphi \psi(\mu) = \varphi(A_\mu) = \chi_{A_\mu} = \mu \) by Theorem 3.25. Therefore \( \varphi \) and \( \psi \) are inverses of each other.

From Theorem 3.30 we obtain the following theorem.

Theorem 3.31. There is a one-to-one correspondence between the set of maximal ideals of a pseudo MV-algebra \( A \) and the set of fuzzy maximal ideals \( \mu \) of \( A \) such that \( \mu(0) = 1 \) and \( \mu(1) = 0 \).

Remark 3.32. Theorem 3.31 implies Theorem 3.22 of [7], the analogous one for MV-algebras.
Acknowledgements

The author thanks Professor A. Walendziak for his helpful comments.

References


Received 23 February 2007
Revised 4 April 2007