VErTEX-DOMINATING CYCLES
IN 2-CONNECTED BIPARTITE GRAPHS

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Abstract

A cycle $C$ is a vertex-dominating cycle if every vertex is adjacent to some vertex of $C$. Bondy and Fan [4] showed that if $G$ is a 2-connected graph with $\delta(G) \geq \frac{1}{3}(|V(G)| - 4)$, then $G$ has a vertex-dominating cycle. In this paper, we prove that if $G$ is a 2-connected bipartite graph with partite sets $V_1$ and $V_2$ such that $\delta(G) \geq \frac{1}{3}(\max\{|V_1|, |V_2|\} + 1)$, then $G$ has a vertex-dominating cycle.

Keywords: vertex-dominating cycle, dominating cycle, bipartite graph.

2000 Mathematics Subject Classification: 05C38, 05C45.

1. Introduction

In this paper, we only consider finite undirected graphs without loops or multiple edges. We denote the degree of a vertex $x$ in a graph $G$ by $d_G(x)$. Let $\delta(G)$ be the minimum degree of a graph $G$. We denote the number of components of $G$ by $\omega(G)$. A connected graph $G$ is defined to be $t$-tough if $|S| \geq t \cdot \omega(G - S)$ for every cutset $S$ of $V(G)$. The toughness of $G$, denoted by $t(G)$, is the maximum value of $t$ for which $G$ is $t$-tough (taking $t(K_n) = \infty$ for all $n \geq 1$). A set $S$ of vertices in a graph $G$ is said to be $d$-stable if the distance of each pair of distinct vertices in $S$ is at least $d$.

In 1960, Ore introduced a degree sum condition for hamiltonian cycles.

Theorem 1 (Ore [8]). Let $G$ be a graph on $n \geq 3$ vertices. If $d_G(x) + d_G(y) \geq n$ for any nonadjacent vertices $x$ and $y$, then $G$ is hamiltonian.
It is observed that weaker conditions guarantee the existence of hamiltonian cycles by putting a further assumption on graphs. For example, Jung (1972) and Moon and Moser (1963) showed that weaker degree sum conditions guarantee hamiltonian cycles in 1-tough graphs and in bipartite graphs, respectively.

**Theorem 2** (Jung [6]). Let $G$ be a 1-tough graph of order $n \geq 11$. If $d_G(x) + d_G(y) \geq n - 4$ for any nonadjacent vertices $x$ and $y$, then $G$ is hamiltonian.

**Theorem 3** (Moon and Moser [7]). Let $G$ be a bipartite graph with partite sets $V_1$ and $V_2$, where $|V_1| = |V_2| = n$. If $d_G(x) + d_G(y) \geq n + 1$ for each pair of nonadjacent vertices $x \in V_1$ and $y \in V_2$, then $G$ is hamiltonian.

A cycle $C$ is a dominating cycle if every edge is incident with some vertex of $C$. A cycle $C$ is called a vertex-dominating cycle if every vertex is adjacent to some vertex of $C$. A dominating cycle is can be consider as a generalization of a hamiltonian cycle, and a vertex-dominating cycle as a generalization of a dominating cycle. Therefore there may be weaker sufficient conditions for the existence of dominating cycles or vertex-dominating cycles which correspond to that for hamiltonicity.

Bondy (1980) and Bondy and Fan (1987) gave a degree sum condition for dominating cycles and vertex-dominating cycles, respectively.

**Theorem 4** (Bondy [3]). Let $G$ be a 2-connected graph on $n$ vertices. If $d_G(x) + d_G(y) + d_G(z) \geq n + 2$ for any independent set of three vertices $x$, $y$ and $z$, then any longest cycle is a dominating cycle.

**Theorem 5** (Bondy and Fan [4]). Let $k \geq 2$ and let $G$ be a $k$-connected graph on $n$ vertices. If $\sum_{x \in S} d_G(x) \geq n - 2k$ for every $3$-stable set $S$ of $G$ of order $k + 1$, then $G$ has a vertex-dominating cycle.

Like hamiltonian cycles, some sufficient conditions for the existence of dominating cycles can be relaxed if we put a further assumption on a graph. In 1989, Bauer, Veldman, Morgana and Schmeichel showed the following result for 1-tough graphs.

**Theorem 6** (Bauer et al. [2]). Let $G$ be a 1-tough graph of order $n$. If $d_G(x) + d_G(y) + d_G(z) \geq n$ for any independent set of three vertices $x$, $y$ and $z$, then any longest cycle in $G$ is a dominating cycle.
In 1984, Ash and Jackson gave a minimum degree condition for a bipartite graph.

**Theorem 7** (Ash and Jackson [1]). Let $G$ be a 2-connected bipartite graph with partite sets $V_1$ and $V_2$, where $\max\{|V_1|, |V_2|\} = n$. If $\delta(G) \geq (n + 3)/3$, then there exists a longest cycle which is a dominating cycle.

In 2003, Saito and the author showed that Theorem 5 also admits a similar relaxation under an additional assumption on toughness.

**Theorem 8** (Saito and Yamashita [9]). Let $k \geq 2$ and $G$ be a $k$-connected graph on $n$ vertices with $t(G) > k/(k + 1)$. If $\sum_{x \in S} d_G(x) \geq n - 2k - 2$ for every 4-stable set $S$ of order $k + 1$, then $G$ has a vertex-dominating cycle.

In this paper, we give a minimum degree condition for a bipartite graph to have a vertex-dominating cycle.

**Theorem 9.** Let $G$ be a 2-connected bipartite graph with partite sets $V_1$ and $V_2$, where $\max\{|V_1|, |V_2|\} = n$. If $\delta(G) \geq (n + 1)/3$, then $G$ has a vertex-dominating cycle.

In Theorem 9, the degree condition is sharp in the following sense. Let $m_i, n_i$ be positive integers, where $1 \leq i \leq 3$. The graph $G$ is obtained from $K_{m_1, n_1} \cup K_{m_2, n_2} \cup K_{m_3, n_3}$, by adding new two vertices $x$ and $y$, and joining both $x$ and $y$ to every vertex in three partite sets of order $n_i$. It is easy to see that $G$ is a 2-connected bipartite graph with partite sets $V_1$ and $V_2$ and $\delta(G) \leq \max\{|V_1|, |V_2|\}/3$, but has no vertex-dominating cycle.

## 2. Proof of Theorem 9

Before proving Theorem 9, we prepare some definitions and notations, and refer to Diestel [5] for terminology and notations not defined here. For a subgraph $H$ of $G$ and a vertex $x \in V(G) - V(H)$, we also denote $N_H(x) := N_G(x) \cap V(H)$ and $d_H(x) := |N_H(x)|$. For $X \subset V(G)$, $N_G(X)$ denote the set of vertices in $G-X$ which are adjacent to some vertex in $X$. Furthermore, for a subgraph $H$ of $G$ and $X \subset V(G) - V(H)$, we sometimes write $N_H(X) := N_G(X) \cap V(H)$. If there is no fear of confusion, we often identify a subgraph $H$ of a graph $G$ with its vertex set $V(H)$. For example, we often write $G - H$ instead of $G - V(H)$. 
We write a cycle $C$ with a given orientation by $\overrightarrow{C}$. For $x, y \in V(C)$, we denote by $C[x, y]$ a path from $x$ to $y$ on $\overrightarrow{C}$. The reverse sequence of $C[x, y]$ is denoted by $\overleftarrow{C}[y, x]$. We define $C(x, y) = C[x, y] - \{x\}$, $C(x, y) = C[x, y] - \{y\}$ and $C(x, y) = C[x, y] - \{x, y\}$. For convenience, we consider $C(x, x) = \emptyset$. For $x \in V(C)$, we denote the successor and the predecessor of $x$ on $\overrightarrow{C}$ by $x^+$ and $x^-$, respectively. A path $P$ connecting $x$ and $y$ is denoted by $P[x, y]$. For a subgraph $H$ of $G$, a path $P[x, y]$ is called an $H$-path if $P[x, y] \cap V(H) = \{x, y\}$ and $E(H) \cap E(P) = \emptyset$.

Let $S$ and $T$ be subsets of $V(G)$. Then $S$ is said to dominate $T$ if every vertex in $T$ either belongs to $S$ or has a neighbor in $S$. If $S$ dominates $V(G)$, then $S$ is called a dominating set.

We define the following sets $F_k$ and $H_k$ of graphs for each odd integer $k \geq 5$. Let $l, b_1, b_2, \ldots, b_l$ be integers with $l \geq 3$ and $b_i \geq (k+1)/2$ ($1 \leq i \leq l$). Let $\bigcup_{i=1}^l K_{(k-3)/2,b_i}$ denote the vertex-disjoint union of $K_{(k-3)/2,b_i}$ for all $i \in \{1, 2, \ldots, l\}$. Then the graph $F_{k,b_1,\ldots, b_l}$ is obtained from $\bigcup_{i=1}^l K_{(k-3)/2,b_i}$ by adding two new vertices $x$ and $y$, and joining both $x$ and $y$ to every vertex of $\bigcup_{i=1}^l K_{(k-3)/2,b_i}$ whose degree in $\bigcup_{i=1}^l K_{(k-3)/2,b_i}$ is $(k-3)/2$. Let $F_k$ be the set of all such graphs. To define $H_k$, let $m, c_1, \ldots, c_m$ be integers at least $(k+1)/2$. The graph $H_{k,c_1,\ldots, c_m}$ is obtained from $\bigcup_{i=1}^m K_{1,c_i}$ by adding $(k-1)/2$ new vertices $x_1, \ldots, x_{(k-1)/2}$, and joining each $x_i$ to every vertex of $\bigcup_{i=1}^m K_{1,c_i}$ whose degree in $\bigcup_{i=1}^m K_{1,c_i}$ is $1$. Let $H_k$ be the set of all such graphs.

To prove Theorem 9, we use the following result due to Wang.

Theorem 10 (Wang [10]). Let $k \geq 2$ and let $G$ be a 2-connected bipartite graph with partite sets $V_1$ and $V_2$. If $d_G(x) + d_G(y) \geq k + 1$ for every pair of nonadjacent vertices $x$ and $y$, then $G$ contains a cycle of length at least
min\{2a,2k\} where \( a = \min\{|V_1|, |V_2|\} \), unless \( 5 \leq k \leq a \), \( k \) is odd and \( G \in \mathcal{F}_k \cup \mathcal{H}_k \).

**Proof of Theorem 9.** Suppose that \( G \) has no vertex-dominating cycle. Let \( C \) be a longest cycle in \( G \) such that \( \omega(G - C) \) is as small as possible, and let \( |V_1| = n_1, |V_2| = n_2 \) and \( n_1 \leq n_2 \).

**Claim 1.** \( |C| = \frac{2}{3}(2n_2 - 1) \) and \( |V_2 - C| = \frac{1}{3}(n_2 + 1) \).

**Proof.** First suppose that \( G \in \mathcal{F}_k \). Since \( \delta(G) = \frac{1}{2}(k + 1) \) and \( l \geq 3 \), we have

\[
\frac{1}{3}(n_2 + 1) = \frac{1}{3} \left( \sum_{i=1}^{l} b_i + 1 \right) \geq \frac{1}{3} \left( \frac{l(k + 1)}{2} + 1 \right) = \frac{l}{3} \delta(G) + \frac{1}{3} > \delta(G).
\]

This contradicts the degree condition. Hence \( G \not\in \mathcal{F}_k \). Next suppose that \( G \in \mathcal{H}_k \). Since \( \delta(G) = \frac{1}{2}(k + 1) \) and \( m \geq 3 \), we get

\[
\frac{1}{3}(n_2 + 1) = \frac{1}{3} \left( \sum_{i=1}^{m} c_i + 1 \right) \geq \frac{1}{3} \left( \frac{m(k + 1)}{2} + 1 \right) = \frac{m}{3} \delta(G) + \frac{1}{3} > \delta(G),
\]

a contradiction. Therefore \( G \not\in \mathcal{H}_k \).

Since \( d_G(x) + d_G(y) \geq \frac{2}{3}(n_2 + 1) = \frac{1}{3}(2n_2 - 1) + 1 \) for any \( x, y \in V(G) \), we obtain \( |C| \geq \min\{2n_1, \frac{2}{3}(2n_2 - 1)\} \) by Theorem 10. Suppose that \( |C| \geq 2n_1 \). Then \( V_1 \subset V(C) \). Since \( G \) is 2-connected, \( N_C(v_2) \neq \emptyset \) for any \( v_2 \in V_2 - C \). Hence \( C \) is a vertex-dominating cycle, a contradiction. Suppose that \( |C| > \frac{2}{3}(2n_2 - 1) \). Then \( |V_1 - C| \leq |V_2 - C| < n_2 - \frac{1}{3}(2n_2 - 1) = \frac{1}{3}(n_2 + 1) \).

Since \( \delta(G) \geq \frac{1}{3}(n_2 + 1) \), \( N_C(v) \neq \emptyset \) for any \( v \in V(G - C) \), that is, \( C \) is a vertex-dominating cycle, a contradiction. Thus we obtain \( |C| = \frac{2}{3}(2n_2 - 1) \) and \( |V_2 - C| = \frac{1}{3}(n_2 + 1) \).

Note that \( \frac{2}{3}(2n_2 - 1) \) and \( \frac{1}{3}(n_2 + 1) \) are integers. We shall partition \( V_i - C \) \((i = 1, 2) \) into three subsets as follows:

\[
X_i := \{ x_i \in V_i - C : N_C(x_i) \neq \emptyset, N_{G-C}(x_i) \neq \emptyset \},
\]

\[
Y_i := \{ y_i \in V_i - C : N_{G-C}(y_i) = \emptyset \} \quad \text{and}
\]

\[
Z_i := \{ z_i \in V_i - C : N_C(z_i) = \emptyset \}.
\]

**Claim 2.** For any \( x_2 \in X_2 \), \( |N_C(x_2)| \geq \frac{1}{3}(n_2 + 1) - (|X_1| + |Z_1|) \geq |Y_1| \).
Proof. By the degree condition, for any \( x_2 \in X_2 \), \(|N_C(x_2)| \geq \delta(G) - (|X_1| + |Z_1|) \geq \frac{1}{3}(n_2 + 1) - (|X_1| + |Z_1|) \). Moreover, it follows from Claim 1 that \(|N_C(x_2)| \geq \frac{1}{3}(n_2 + 1) - (|X_1| + |Z_1|) \geq \frac{1}{3}(n_2 + 1) - (\frac{1}{3}(n_2 + 1) - |Y_1|) \geq |Y_1| \).

Claim 3. Let \( z_i \in Z_i \). Then \( N_G(z_i) = V_{3-i} - C \) and \(|V_{3-i} - C| = \frac{1}{3}(n_2 + 1) \).

Proof. Suppose that \( z_i \in Z_i \). By Claim 1 and the definition of \( Z_i \), \( \frac{1}{3}(n_2 + 1) \geq |V_{3-i} - C| \geq d_G(z_i) \geq \frac{1}{3}(n_2 + 1) \). This implies \(|V_{3-i} - C| = d_G(z_i) = \frac{1}{3}(n_2 + 1) \), and so \( N_G(z_i) = V_{3-i} - C \) and \(|V_{3-i} - C| = \frac{1}{3}(n_2 + 1) \).

Claim 4. \( Z_1 \) or \( Z_2 \) is non-empty. If \( Z_2 \) is not empty, then \(|V_1| = |V_2| \) and \( Y_1 \) is empty.

Proof. If \( Z_1 = \emptyset \) and \( Z_2 = \emptyset \), then \( C \) is a vertex-dominating cycle. Hence \( Z_1 \neq \emptyset \) or \( Z_2 \neq \emptyset \). If \( Z_2 \neq \emptyset \) then, by Claims 1 and 3, \(|V_1 - C| = |V_2 - C| = \frac{1}{3}(n_2 + 1) \), that is, \(|V_1| = |V_2| \). By Claim 3 and the definition of \( Y_1 \), we have \( Y_1 = \emptyset \).

In view of Claim 4 and the symmetry, we may assume in the rest of the proof that \( Z_1 \) is non-empty and consequently \( Y_2 \) is empty.

If \( X_2 = \emptyset \), let \( x_a, x_b \in X_1 \); otherwise let \( x_a \in X_1 \cup X_2 \) and \( x_b \in X_2 \). By Claims 3 and 4, \( X_1 \cup X_2 \cup Z_1 \cup Z_2 \) is contained in a component of \( G - C \). Hence there exists a path \( P_0[x_a, x_b] \) in \( G - C \). We can choose \( x_a, x_b \) such that \( (i) \ a \in N_G(x_a) \) and \( b \in N_G(x_b) \) \( (a \neq b) \) are as close as possible on \( C \), and \( (ii) \ |P_0| \) is as large as possible, subject to \( (i) \). Let \( C_0 = x_aC[b, a]P_0[x_a, x_b] \), \( U_1 := C(b, a) \cap V_i \) and \( U_1' := C(a, b) \cap V_i \). We give an orientation on \( C \) such that \(|C(a, b)| \leq |C(b, a)| \). By the choice of \( x_a \) and \( x_b \), we have

\[
(1) \quad |C(a, b)| \leq \frac{1}{2}|C| - 1 = \frac{1}{3}(2n_2 - 1) - 1 = 2 \left( \frac{1}{3}(n_2 + 1) - 1 \right).
\]

Claim 5. \( C[b, a] \) dominates \( X_1 \cup X_2 \cup Y_1 \cup U_1 \).

Proof. By the choice of \( x_a \) and \( x_b \), \( N_G(x) \cap C(a, b) = \emptyset \) for any \( x \in X_1 \cup X_2 \). Hence \( N_G(x) \cap C[b, a] = \emptyset \) for any \( x \in X_1 \cup X_2 \), and so \( C[b, a] \) dominates \( X_1 \) and \( X_2 \). It follows from \( (1) \) that \(|U_2| \leq \frac{1}{3}(n_2 + 1) - 1 \). Therefore \( N_G(y_1) \cap C[b, a] = \emptyset \) for any \( y_1 \in Y_1 \). Moreover, by the choice of \( x_a \) and \( x_b \), \( N_G(U_1) \cap X_2 = \emptyset \), and so \( N_G(u_1) \cap C[b, a] = \emptyset \) for any \( u_1 \in U_1 \). Hence \( C[b, a] \) dominates \( Y_1 \) and \( U_1 \).
Case 1. $|C(a, b)|$ is even.
Then $x_a \in X_1$ and $x_b \in X_2$. By Claim 3, $\{x_a, x_b\}$ dominates $Z_1$ and $Z_2$. Hence if $C_0$ dominates $U_2$ then by Claim 5, $C_0$ is a vertex-dominating cycle. Thus, we may assume that $C_0$ does not dominate $U_2$, that is, there exists $u_2 \in U_2$ such that $N_G(u_2) \subset U_1 \cup Y_1$. By the degree condition, we have
\[
\frac{1}{3}(n_2 + 1) \leq d_G(u_2) \leq |U_1| + |Y_1| \leq \frac{1}{2}|C(a, b)| + |Y_1|,
\]
and by Claim 1,
\[
|C| = \frac{3}{2}(2n_2 - 1) \leq 2|C(a, b)| + 4|Y_1| - 2.
\]
By combining (1) and (2), we have $|Y_1| \geq 1$. Assume that $|Y_1| \geq 2$. Since $u_2 \neq b^\prime$, $|C(a, b)| \geq 4$. It follows from Claim 2 and (3) that
\[
(|N_G(X_2)| + 1)(|C(a, b)| + 1) - |C| 
\geq (|Y_1| + 1)(|C(a, b)| + 1) - 2|C(a, b)| + 4|Y_1| - 2
\]
\[
= (|Y_1| - 1)(|C(a, b)| - 3) > 0,
\]
and so $(|N_G(X_2)| + 1)(|C(a, b)| + 1) > |C|$. On the other hand, by the choice of $x_a$ and $x_b$, $C - N_G(\{x_a\} \cup X_2)$ consists of at least $|N_G(X_2)| + 1$ paths of order at least $|C(a, b)|$. This implies $|C| \geq (|N_G(X_2)| + 1)(|C(a, b)| + 1)$. Thus we get a contradiction.

Hence $|Y_1| = 1$, say $y_1 \in Y_1$. By (1) and (2), $|C(a, b)| = |C(b, a)| = 2\left(\frac{1}{3}(n_2 + 1) - 1\right)$. Therefore $N_G(X_1 \cup X_2) = \{a, b\}$, and so $\{a, b\}$ dominates $X_1$ and $X_2$. By using the same argument as the proof of Claim 5, $C[a, b]$ dominates $U_1$ and $Y_1$. Hence there exists $u_2' \in U_2'$ such that $N_G(u_2') \subset U_1 \cup Y_1$, otherwise $x_aC[a, b]x_bP_3x_a$ is a vertex-dominating cycle. Since $|U_1| = |U_1'| = \frac{1}{3}(n_2 + 1) - 1$, we see that $y_1 \in N_G(u_2)$ and $y_1 \in N_G(u_2')$.

Let $v_2' \in C(a, u_2')$ and $v_2 \in C(b, u_2)$ such that (i) $y_1 \in N_G(v_2')$ and $y_1 \in N_G(v_2)$ and (ii) $C(a, v_2') \cup C(b, v_2)$ is inclusion-minimal, subject to (i). By the existence of the C-path $v_2y_1v_2'$, there exists a C-path $P_1[w_2, w_2']$ joining $C(b, v_2)$ and $C(a, v_2')$. Choose $P_1$ such that $C(a, v_2') \cup C(b, v_2)$ is inclusion-minimal. By the choice of $v_2$ and $P_1$, $N(w) \cap (Y_1 \cup C(b, w_2)) = \emptyset$ for any $w \in C(a, w_2')$. Thus, since $|C(a, w_2')| \leq |C(a, b)| \leq 2\left(\frac{1}{3}(n_2 + 1) - 1\right)$, $N(w) \cap (C[w_2', b] \cup C[w_2, a]) \neq \emptyset$ for any $w \in C(a, w_2')$. Hence $C[w_2', b] \cup C[w_2, a]$ dominates $C[a, w_2']$. Similarly, $C[w_2', b] \cup C[w_2, a]$ dominates $C[b, w_2')$. Moreover, since $u_2 \in C[b, w_2'] \cup C[w_2, a]$, $C[w_2', b] \cup C[w_2, a]$ dominates $Y_1$. Hence
$x_a \overrightarrow{C}[a, w_2] P_1[w_2, w'_2] C(w'_2, b] P_b[x_b, x_a]$ is a vertex-dominating cycle. This completes the proof of Case 1.

**Case 2.** $|C(a, b)|$ is odd.

Note that $x_a \in X_i$ and $x_b \in X_i$ for $i = 1$ or $i = 2$.

**Case 2.1.** $Z_2 = \emptyset$.

Then $X_2 \neq \emptyset$ and $|X_2| = \frac{1}{3}(n_2 + 1)$, otherwise $C$ is a hamiltonian cycle by Claim 4. By the choice of $x_a$ and $x_b$, note that $x_a, x_b \in X_2$. By Claim 3, $\{x_a, x_b\}$ dominates $Z_1$. Hence there exists $u_2 \in U_2$ such that $N_G(u_2) \subset U_1 \cup Y_1$, otherwise $C_0$ is a vertex-dominating cycle. Since $u_2 \neq a^+, b^-$, we have

(4) $|C(a, b)| \geq 5$.

Since $a^+, b^- \in V_2$ and $|C(a, b)|$ is odd,

(5) $\frac{1}{3}(n_2 + 1) \leq d_G(u_2) \leq |U_1| + |Y_1| \leq \frac{1}{2}(|C(a, b)| - 1) + |Y_1|,$

and by Claim 1,

(6) $|C| = \frac{2}{3}(2n_2 - 1) \leq 2|C(a, b)| + 4|Y_1| - 4.$

By (1) and (5), we have $|Y_1| \geq 2$. Since $C - N_C(X_2)$ has at least $|N_C(X_2)|$ paths of order at least $|C(a, b)|$, we have $|C| \geq |N_C(X_2)||C(a, b)| + 1$.

Assume that $|Y_1| \geq 4$. It follows from Claim 2, (4) and (6) that

$$|N_C(X_2)|||C(a, b)| + 1| - |C| \geq |Y_1|||C(a, b)| + 1| - (2|C(a, b)| + 4|Y_1| - 4) = (|Y_1| - 2)(||C(a, b)| - 3) - 2 > 0,$$

a contradiction. Therefore $|Y_1| = 2$ or $|Y_1| = 3$.

**Claim 6.** (i) $X_1 = \emptyset$,

(ii) $|Z_1| = \frac{1}{3}(n_2 + 1) - |Y_1|$ and

(iii) $N_C(X_2) = N_C(x_2)$ for any $x_2 \in X_2$. 
**Proof.** First, suppose that $X_1 \neq \emptyset$, say $x_1 \in X_1$. Since $C - N_C(\{x_1\} \cup X_2)$ has at least $|N_C(X_2)| + 1$ paths of order at least $|C(a, b)|$, $|C| \geq |N_C(\{x_1\} \cup X_2)|(|C(a, b)| + 1)$. By Claim 2, (4) and (6),

$$|N_C(\{x_1\} \cup X_2)|(|C(a, b)| + 1) - |C| \geq (|Y_1| + 1)(|C(a, b)| + 1) - (2|C(a, b)| + 4|Y_1| - 4) = (|Y_1| - 1)(|C(a, b)| - 3) + 2 > 0,$$

a contradiction. Next suppose that $|Z_1| < \frac{1}{2}(n_2 + 1) - |Y_1|$ or $N_C(X_2) > N_C(x_2)$ for some $x_2 \in X_2$. Then, by Claim 2, $|N_C(X_2)| \geq |Y_1| + 1$. By a similar argument as above, we obtain a contradiction. 

Since $|Y_1| \geq 2$, we have $|X_2| \geq 2$ and by Claim 6 (iii), we can choose $x_a, x_b$ with $x_a \neq x_b$. By Claim 3 and Claims 6 (i) and (ii), we obtain $|P_0| = |X_2| + |Z_1| - |Y_1| + 1 = \frac{2}{3}(n_2 + 1) - 2|Y_1| + 1$. On the other hand, by (1) and (5), $|C(a, b)| = \frac{2}{3}(n_2 + 1) - 2|Y_1| + 1$. Hence $C_0$ and $C$ have the same length. Since $C(a, b) \cup Y_1$ is contained in a component of $G - C_0$ and $|X_2 - P_0| = |Y_1| - 1$, we have $\omega(G - C_0) = |Y_1|$. Note that $\omega(G - C) = |Y_1| + 1$. Therefore $\omega(G - C) > \omega(G - C_0)$. This contradicts the choice of $C$.

**Case 2.2.** $Z_2 \neq \emptyset$.

Then $Y_1 = \emptyset$ by Claim 3. Since $|U_1| \leq \frac{1}{3}(n_2 + 1) - 1$, $N(u_2) \cap C[b, a] \neq \emptyset$ for any $u_2 \in U_2$, that is, $C[b, a]$ dominates $U_2$. Suppose that $x_a \neq x_b$. By Claim 3, $P_0[x_a, x_b]$ dominates $Z_1$ and $Z_2$, and so $C_0$ is a vertex-dominating cycle. Therefore $x_a = x_b$. By the 2-connectivity of $G$ and the choice of $x_a$ and $x_b$, there exists $x_d \in X_1 \cup X_2$ such that $x_d \neq x_a$ and $N_C(x_d) \cap C(b, a) \neq \emptyset$, say $d \in N_C(x_d) \cap C(b, a)$. Choose $x_d$ such that $\min\{|C(b, d)|, |C(d, a)|\}$ as small as possible. Without loss of generality, we may assume that $|C(b, d)| \geq |C(d, a)|$. By the choice of $x_d$, $C[a, d]$ dominates $X_1$ and $X_2$. By Claim 3, there exists a path $P_3[x_a, x_d]$ in $G - C$, which dominates $Z_1$ and $Z_2$. Since $|C[a, b]| \geq 3$, we have $|C(d, a)| \leq \frac{1}{3}|C| - 1 \leq 2(\frac{1}{3}(n_2 + 1) - 1) - 1$. Since $|C(a, d) \cap V_1|, |C(d, a) \cap V_2| \leq \frac{1}{3}(n_2 + 1) - 1$ and $Y_1 = Y_2 = \emptyset$, we can see that $C[a, d]$ dominates $C(d, a)$. Hence $x_aC[a, d]P_3[x_d, x_a]$ is a vertex-dominating cycle. This completes the proof of Case 2.2 and the proof of Theorem 9. 

**References**


Received 28 April 2006
Revised 23 February 2007
Accepted 23 February 2007