DIGRAPHS WITH ISOMORPHIC UNDERLYING AND DOMINATION GRAPHS: CONNECTED $UG^C(D)$

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Dedicated to the memory of Kenneth P. Bogart

Abstract

The domination graph of a directed graph has an edge between vertices $x$ and $y$ provided either $(x, z)$ or $(y, z)$ is an arc for every vertex $z$ distinct from $x$ and $y$. We consider directed graphs $D$ for which the domination graph of $D$ is isomorphic to the underlying graph of $D$. We demonstrate that the complement of the underlying graph must have $k$ connected components isomorphic to complete graphs, paths, or cycles. A complete characterization of directed graphs where $k = 1$ is presented.

Keywords: domination graph, domination, graph isomorphism, underlying graph.

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1. Introduction

Domination graphs were first introduced by Fisher, Lundgren, Merz and Reid [15] to describe the structure of the domination graphs and competition graphs of tournaments. Further refinements were made on these characterizations for tournaments in later work, including Cho, Doherty, Kim, and Lundgren [4, 5], and Fisher et al. [11, 12, 13, 14, 16], but the characterization of the structure of domination graphs of arbitrary digraphs has proved elusive. Here we will examine digraphs $D$ with the property that the underlying graph of $D$ is isomorphic to its domination graph.

A directed graph $D = (V, A)$ will consist of a nonempty set of vertices $V$ and a set of ordered pairs of vertices $A$. We do not permit loops, but we will allow for bidirectional arcs: That is, the pair of arcs $(x, y), (y, x)$. The complement of $D$, $D^c$ has the vertex set $V$ and the set of ordered pairs not in $A$, although we still exclude loops. The underlying graph of $D$, $UG(D)$ has the same set of vertices with the set of edges $\{x, y\}$ where either $(x, y)$ or $(y, x)$ is in $A$. If $D$ has no bidirectional arcs then $D$ is an orientation of $UG(D)$.

The union of two graphs or directed graphs is the graph formed by the union of their vertices as well as their sets of edges or arcs. The join of two graphs $G$ and $H$, $G + H$, is the graph that consists of $G \cup H$ and all edges joining a vertex in $G$ and a vertex in $H$. We extend this definition to directed graphs as follows. The join of $D_1$ and $D_2$ consists of $D_1 \cup D_2$ together with all bidirectional arcs between any vertex of $D_1$ and any vertex of $D_2$.

The study of domination graphs in tournaments arose from the goal of characterizing competition graphs in tournaments. A competition graph of $D$ has vertex set $V$ with an edge between $x$ and $y$ if and only if there is a third vertex $z$ with both arcs $(x, z)$ and $(y, z)$ in $A$. Complementarily, a pair of vertices $x, y$ in $D$ are a dominating pair if and only if for all other vertices $z$, either $(x, z)$ is an arc or $(y, z)$ is an arc. The domination graph of $D$, $dom(D)$ is the undirected graph consisting of vertex set $V$ with an edge between every dominating pair.

In the second section of this paper, previous results are presented that build a foundation for the current work. We then examine the structural necessities in the underlying graph given that we have isomorphic underlying and domination graphs. Finally, a complete characterization of
digraphs with $UG(D) \cong dom(D)$ is developed where the complement of the underlying graph, $UG^c(D)$ is connected.

![Figure 1. The join of two directed graphs.](image)

**2. Symmetric and Antisymmetric Digraphs**

The underlying graph of a tournament is the complete graph on $n$ vertices, $K_n$. For tournaments on more than three vertices, $dom(D)$ is not equal to $K_n$. A *semicomplete digraph* is a tournament with the possible addition of bidirectional arcs. Factor and Factor [8] characterize semicomplete digraphs which have domination graph equal to $K_n$. Factor and Langley [9] were able to fully characterize digraphs $D$ with $UG(D) = dom(D)$. The underlying graph of such a digraph is a complete multipartite graph.

If $D$ is an orientation of $UG(D)$ then $D$ is also called an *antisymmetric* digraph. Bergstrand and Friedler [2] show that if $dom(D)$ is isomorphic to $UG(D)$ for an antisymmetric digraph, then $UG(D)$ must be a star or a collection of independent vertices.

If every arc in $D$ is bidirectional, then $D$ is a *symmetric* digraph. The characterization of $UG(D) \cong dom(D)$ for symmetric digraphs follows from a number of equivalences.

Notice that in a tournament, $D^c$ is the reversal of $D$. That is, create a new tournament by reversing every arc. Through this definition for tourna-
ments, Fisher et al. show that the competition graph of $D$ is isomorphic to the domination graph of $D^c$ [15].

The neighborhood graph or two-step graph, $N(G)$, of a graph $G$ has an edge between vertices $x$ and $y$ provided both $\{x, z\}$ and $\{y, z\}$ exist in $G$ for some third vertex $z$. Lundgren, Maybee and Rasmussen [18, 19] show that the competition graph of a symmetric digraph $D$ is the neighborhood graph of $UG(D)$. Finally, Brigham and Dutton [3] characterize all graphs isomorphic to their own neighborhood graphs as follows:

**Theorem 2.1** (Brigham and Dutton [3]). $N(G) \cong G$ if and only if every component of $G$ is either an odd cycle or a complete graph having other than two nodes.

Following this chain of equivalences we have the following theorem,

**Theorem 2.2** (Factor and Langley [9]). Let $D$ be a symmetric digraph. Then $UG(D) \cong dom(D)$ if and only if $UG(D)$ is the join of independent sets with other than two vertices and components that are the complements of odd cycles.

For example, if $C_n$ is the cycle on $n$ vertices, where $n \geq 3$ and $n$ is odd, by making $D$ the complete biorientation of $C_n^c$, we obtain $UG(D) \cong dom(D)$.

In the following section we will consider what happens when some edges of an underlying graph are bioriented and some are not.

### 3. The Structure of the Underlying Graph

It is the nature of an underlying graph of a digraph $D$ that $UG(D)$ will have many edges when $UG(D) \cong dom(D)$. This makes its complement a more desirable structure with which to work and to express results. Therefore, we refer to both the graphs of $UG(D)$ and $dom(D)$ as well as their complements throughout the course of this paper.

To begin, we show that if we have a set of underlying graphs that are isomorphic to their associated domination graphs, then their join retains that isomorphic property.

**Theorem 3.1.** If $D_1, \ldots, D_k$ are directed graphs such that $UG(D_i) \cong dom(D_i)$ for $i = 1, \ldots, k$ and $D = D_1 + D_2 + \cdots + D_k$, then $UG(D) \cong dom(D)$. Also
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1. \( UG(D) = \sum_{i=1}^{k} UG(D_i) \),
2. \( \text{dom}(D) = \sum_{i=1}^{k} \text{dom}(D_i) \),
3. \( UG^c(D) = \bigcup_{i=1}^{k} UG^c(D_i) \),
4. \( \text{dom}^c(D) = \bigcup_{i=1}^{k} \text{dom}^c(D_i) \).

**Proof.** Let \( D_1, \ldots, D_k \) be directed graphs such that \( UG(D_i) \cong \text{dom}(D_i) \) for \( i = 1, \ldots, k \). Let \( D = \sum_{i=1}^{k} D_i \). Formula 1 follows directly from the definition of the join. Suppose \( x \in D_i \) and \( y \in D_j \). If \( i = j \), and \( x \) and \( y \) are a dominating pair in \( D_i \), then one or the other dominates every vertex in \( D_i \). By the construction of the join, \( x \) and \( y \) dominate every vertex that is not in \( D_i \), so they remain a dominating pair in \( D \). If \( x \) and \( y \) are not a dominating pair, there must be some vertex \( z \) in \( D_i \) that both fail to dominate. They will not dominate \( z \) in the join, so they are not a dominating pair in \( D \). If \( i \neq j \), \( x \) dominates \( D_j \), \( y \) dominates \( D_i \) and both dominate the remaining vertices, so \( x \) and \( y \) form a dominating pair in \( D \). Since both formula 1 and 2 hold, it follows that \( UG(D) \cong \text{dom}(D) \). Finally, formulas 3 and 4 follow immediately from the definition of complement. 

The remainder of the results in this section builds the types of components that are possible in the four graph structures. Specifically, we discuss possible adjacencies of the vertices. These adjacencies lead to cycles in the complements, independent sets in the underlying and domination graphs, and the corresponding cliques in \( UG^c(D) \) and \( \text{dom}^c(D) \). Using this foundation, the concluding theorem characterizes the possible structures for a graph where \( UG(D) \) is isomorphic to \( \text{dom}(D) \).

**Lemma 3.2.** If a vertex \( y \) is adjacent to \( x_1, x_2, \ldots, x_r \) in \( UG^c(D) \) where \( r \geq 2 \), then \( x_i \) and \( x_j \) are adjacent in \( \text{dom}^c(D) \), for all \( 1 \leq i < j \leq r \).

**Proof.** Vertex \( y \) is adjacent to \( x_1, \ldots, x_r \) in \( UG^c(D) \) so \( y \) is not adjacent to those vertices in \( UG(D) \). Thus there is no orientation of \( UG(D) \) that allows \( x_i \) and \( x_j \) to dominate \( y \). Since \( x_i \) and \( x_j \) are not adjacent in \( \text{dom}(D) \), they must be adjacent in \( \text{dom}^c(D) \). 

**Corollary 3.3.** If a vertex \( y \) is not adjacent to \( x_1, \ldots, x_r \) in \( UG(D) \) where \( r \geq 2 \), then \( x_i \) and \( x_j \) are not adjacent in \( \text{dom}(D) \), for all \( 1 \leq i < j \leq r \).

**Lemma 3.4.** If \( x_1, x_2, x_3, \ldots, x_r, x_1 \) is an odd length cycle in \( UG^c(D) \) then \( x_1, x_3, \ldots, x_r, x_2, x_4, \ldots, x_{r-1}, x_1 \) is an odd length cycle in \( \text{dom}^c(D) \).
Proof. Consider vertices $x_i, x_{i+1}, x_{i+2}$ of the cycle. By Lemma 3.2, $x_i, x_{i+2}$ are adjacent in $\text{dom}^c(D)$. Since $r$ is odd, we obtain the cycle $x_1, x_3, \ldots, x_r, x_2, x_4, \ldots, x_{r-1}, x_1$ in $\text{dom}^c(D)$. 

Corollary 3.5. If $x_1, x_2, x_3$ is a 3-cycle in $UG^c(D)$, then $x_1, x_2, x_3$ is a 3-cycle in $\text{dom}^c(D)$.

Corollary 3.6. If $x_1, x_2, x_3$ is an independent set in $UG(D)$, then $x_1, x_2, x_3$ is an independent set in $\text{dom}(D)$.

Lemma 3.7. Let $UG(D)$ be isomorphic to $\text{dom}(D)$ and $r$ be an integer, $r \geq 3$. Then $x_1, x_2, \ldots, x_r$ are independent in $\text{dom}(D)$ if and only if $x_1, x_2, \ldots, x_r$ are independent in $UG(D)$.

Proof. By Corollary 3.6, if $x_1, x_2, x_3$ are independent in $UG(D)$ then $x_1, x_2, x_3$ are independent in $\text{dom}(D)$, so the set of independent triples of $UG(D)$ is a subset of the set of independent triples of $\text{dom}(D)$. Since $UG(D)$ is isomorphic to $\text{dom}(D)$, the number of independent triples of vertices is the same in both graphs. Therefore, the sets of independent triples must be the same as well. An independent set of vertices is completely determined by the independent triples contained within it.

Corollary 3.8. Let $UG(D)$ be isomorphic to $\text{dom}(D)$ and $r \geq 3$. Then $x_1, x_2, \ldots, x_r$ form a clique in $UG^c(D)$ if and only if $x_1, x_2, \ldots, x_r$ form a clique in $\text{dom}^c(D)$.

Lemma 3.9. If $UG(D)$ is isomorphic to $\text{dom}(D)$ and there is no edge between $y$ and any of $x_1, x_2, \ldots, x_r$, $r \geq 3$ in $UG(D)$, then $y, x_1, x_2, \ldots, x_r$ are independent in $UG(D)$ and $\text{dom}(D)$, and form a clique in $UG^c(D)$ and $\text{dom}^c(D)$.

Proof. It follows from Corollary 3.3 that $x_1, x_2, \ldots, x_r$ are an independent set in $\text{dom}(D)$ and thus by Lemma 3.7 must be independent in $UG(D)$. Since there are no arcs between $y$ and $x_i$, $y$ can be added to this independent set in $UG(D)$ and hence in $\text{dom}(D)$. Thus they form a clique in $UG^c(D)$ and $\text{dom}^c(D)$ as well.

Lemma 3.10. If $UG(D)$ is isomorphic to $\text{dom}(D)$ and $x_1, x_2, \ldots, x_r$, $r \geq 3$ form a maximal clique in $UG^c(D)$, then $x_1, x_2, \ldots, x_r$ form a connected component isomorphic to $K_r$ in $UG^c(D)$.
Proof. Let $x_1, x_2, \ldots, x_r$, $r \geq 3$ form a maximal clique in $UG^c(D)$. Suppose $y$ is a vertex such that $y \neq x_i$ for $i = 1, \ldots, r$ and there is an edge between $y$ and $x_j$ in $UG^c(D)$. Without loss of generality, let $j = 1$. Then there is no edge between $x_1$ and $y, x_2, \ldots, x_r$ in $UG(D)$, so by Lemma 3.9, $y, x_1, x_2, \ldots, x_r$ form an independent set in $UG(D)$, contradicting the maximality assumption. Therefore, there are no edges between any vertex $x_i$ and any vertex $y$ not in the clique. Thus the clique is a connected component isomorphic to $K_r$ in $UG^c(D)$.

Theorem 3.11. If $UG(D)$ is isomorphic to $dom(D)$, then $UG^c(D)$ is comprised of one or more connected components, each either a complete graph, a path, or a cycle.

Proof. It follows from Lemma 3.9 that any vertex of degree $r \geq 3$ or more in $UG^c(D)$ must be in a clique of 4 or more vertices and consequently, by Lemma 3.10 in a component isomorphic to $K_{r+1}$. So every vertex in $UG^c(D)$ that isn’t in a component isomorphic to $K_r$ must be of degree 2 or less. It follows that every component of $UG^c(D)$ is isomorphic to a complete graph, path, or cycle.

Corollary 3.12. If $UG(D)$ is isomorphic to $dom(D)$, then $UG(D)$ is the join of one or more independent sets, complements of paths, and complements of cycles.

Since it is now shown that if $UG(D) \cong dom(D)$, each component of $UG^c(D)$ must be a complete graph, a cycle or path, we consider the question of which paths, cycles or complete graphs may actually be components of $UG^c(D)$. In partial answer to this question, Theorem 2.1 demonstrates that a complete biorientation of an underlying graph whose complement has components consisting of complete graphs on other than 2 vertices and odd cycles will work. What remains unanswered is whether there exist biorientations of $UG(D)$ where $K_2$, even cycles, and paths are structures which may be components of $UG^c(D)$.

In [9], the authors of the present paper find that a digraph $D$ exists where $UG(D) = dom(D)$ and $K_2$ is a component of $UG^c(D)$.

Theorem 3.13 ([9]). A biorientation $D$ of a graph $G$ on $n \geq 3$ vertices exists such that $UG(D) = dom(D)$ if and only if
1. \( G = \sum_{i=1}^{p} G_i \) where \( G_i, i = 1, \ldots, p - 1 \) are independent sets and 
\( G_p = K_m \) for some \( m \geq 0 \), and
2. if we let \( s \) be the number of independent sets of size 2, then \( s \leq m \).

Theorem 3.13 guarantees that there are digraphs with \( UG^{c}(D) \) containing \( K_2 \), where \( UG(D) = dom(D) \) and, thus \( UG(D) \cong dom(D) \).

Similarly for even cycles of the form \( C_{2i} \), \( i \geq 2 \), we are able to construct digraphs where \( UG(D) \cong dom(D) \) and \( C_{2i} \) is a component of \( UG^{c}(D) \). However, not all graphs whose complements contain these cycles as components have a biorientation yielding \( UG(D) \cong dom(D) \). As we see in the following lemma, the existence of an even length cycle on more than 4 vertices as a component in \( UG^{c}(D) \) necessarily requires smaller cycles.

**Lemma 3.14.** If \( UG^{c}(D) \) contains a component isomorphic to a cycle on an even number of vertices, \( C_{2k} \) where \( k \geq 3 \), then \( dom^{c}(D) \) contains at least two cycles of length \( k \).

**Proof.** Let \( x_1, x_2, x_3, \ldots, x_{2k}, x_1 \) form a cycle in \( UG^{c}(D) \). By Lemma 3.2 \( x_i \) and \( x_{i+2 \mod 2k} \) will be adjacent in \( dom^{c}(D) \). This means that \( x_1, x_3, \ldots, x_{2k-1}, x_1 \) and \( x_2, x_4, \ldots, x_{2k}, x_2 \) form two cycles of length \( k \) in \( dom^{c}(D) \). ■

For existence of digraphs \( D_i \) where \( C_{2i} \) is a component of \( UG^{c}(D_i) \) and \( UG(D) \cong dom(D) \), consider the following construction.

Define \( D_i \) to be a digraph with \((i - 1)2^i + 2^{i-1}\) vertices. Begin by constructing \( UG^{c}(D) \). Choose \( 2^i \) vertices to form one cycle of length \( 2^i \) in \( UG^{c}(D_i) \); \( 2^i \) vertices to form two cycles of length \( 2^{i-1} \) in \( UG^{c}(D_i) \) and so on, down to \( 2^{i-3} \) cycles of length \( 2^3 \). This leaves \( 2^i + 2^{i-1} \) vertices. Of these vertices, \( 2^i \) will be split into 4-cycles in \( UG^{c}(D_i) \) and the rest will
be isolated vertices. For $1 \leq j \leq 4$ and $1 \leq k \leq 2^i - 2$, label $2^i$ of the remaining vertices $x_{j,k}$, and label the last $2^i - 1$ vertices $y_{1,k}$, $y_{2,k}$. Form the 4-cycles in $UG^c(D_i)$ as $x_{1,k}, x_{2,k}, x_{3,k}, x_{4,k}, x_{1,k}$ for $k = 1, \ldots, 2^i - 2$. Of course any edge between two vertices in $UG^c(D_i)$ means there will be no arcs between those vertices in $D_i$. Next, to continue the construction of $D_i$, place single arcs $(y_{1,k}, x_{2,k}), (y_{1,k}, x_{1,k+1}), (y_{2,k}, x_{3,k}),$ and $(y_{2,k}, x_{4,k+1})$, for $k = 1, \ldots, 2^i - 2 - 1$ as well as the single arcs $(y_{1,2^{i-2}}, x_{1,1}), (y_{1,2^{i-2}}, x_{4,1}), (y_{2,2^{i-2}}, x_{2,2^{i-2}}),$ and $(y_{2,2^{i-2}}, x_{3,2^{i-2}})$. Place double arcs between all pairs of vertices not otherwise accounted for, so $UG^c(D_i)$ contains one cycle of length $2^i$, two cycles of length $2^{i-1}$ and so on down to $2^{i-2}$ cycles of length 4, and $2^{i-1}$ components isomorphic to $K_1$. By the proof of Lemma 3.14 each cycle of length $2^r$, $r \geq 3$ in $UG^c(D_i)$ will form two cycles of length $2^{r-1}$ in $dom^c(D_i)$. So, $dom^c(D_i)$ will contain the correct number of cycles of length less than $2^i$. The collection of vertices $x_{j,k}$, $y_{1,k}$, $y_{2,k}$ will form the missing cycle of length $2^i$ on the $x_{j,k}$ and the $y_{1,k}, y_{2,k}$ will remain components isomorphic to $K_1$ in $dom^c(D_i)$. We illustrate the construction of $D_3$ in Figure 3 where $C_{25}$ is a component of $UG^c(D_3)$. Dotted lines represent pairs of vertices with no arcs between them, hence the dotted lines are edges in $UG^c(D_3)$. Single arcs are shown. All other pairs of vertices have double arcs, but these are not shown in the figure. The vertices $v_1, v_2, \ldots, v_8, v_1$ form the single 8-cycle in $UG^c(D_3)$. The remaining vertices are labeled as in the construction to form two 4-cycles and four independent vertices in $UG^c(D_3)$.

![Figure 3. Construction of $D_3$.](image)

The graph $dom^c(D_3)$ is not shown, but it will contain two components isomorphic to 4-cycles: $v_1, v_3, v_5, v_7, v_1$ and $v_2, v_4, v_6, v_8, v_2$ and one 8-cycle: $x_{1,1}, x_{3,1}, x_{4,2}, x_{2,2}, x_{3,2}, x_{1,2}, x_{2,1}, x_{4,1}, x_{1,1}$. The vertices $y_{1,1}, y_{1,2}, y_{2,1},$ and $y_{2,2}$ will remain independent in $dom^c(D_3)$ as well, so $dom^c(D_3) \cong UG^c(D_3)$. 


To conclude our discussion on cycles for this paper, we now consider cycles \( C_r, r = 2^l k \), where \( k \geq 3 \) is an odd integer, and \( l \) is a positive integer. Cycles of this form cannot be components of \( UG^c(D) \) when \( UG(D) \cong dom(D) \).

**Lemma 3.15.** If \( UG(D) \cong dom(D) \), and \( x_1, \ldots, x_r \) form an odd length cycle in \( dom^c(D) \), then \( x_1, \ldots, x_r \) are vertices of an odd length cycle in \( UG^c(D) \).

**Proof.** From Lemma 3.4, every odd length cycle in \( UG^c(D) \) generates an odd length cycle on the same set of vertices in \( dom^c(D) \). If \( UG^c(D) \cong dom^c(D) \), each graph must contain the same number of odd cycles. Thus, there can be no odd length cycles that are not generated as described in Lemma 3.4.

**Theorem 3.16.** Let \( r = 2^l k \) where \( k \geq 3 \) is an odd integer and \( l \) is a positive integer. If \( UG(D) \) is isomorphic to \( dom(D) \), then no component of \( UG^c(D) \) is isomorphic to \( C_r \), a cycle of length \( r \).

**Proof.** This follows by induction on \( l \). Suppose \( l = 1 \). Then \( r = 2k \) where \( k \) is odd. Suppose \( x_1, x_2, x_3, \ldots, x_r, x_1 \) form a component isomorphic to \( C_r \) in \( UG^c(D) \). Consider the indices mod \( r \). Since \( r \) is even, by the proof of Lemma 3.14 \( x_1, x_3, \ldots, x_{r-1}, x_1 \) form a cycle of length \( k \) in \( dom^c(D) \), (and \( x_2, x_4, \ldots, x_r, x_2 \) form a cycle of length \( k \) in \( dom^c(D) \) as well). However, it follows by Lemma 3.15 that \( x_1, x_3, \ldots, x_{r-1}, x_1 \) are vertices of an odd cycle in \( UG^c(D) \) which contradicts the fact that these vertices are part of a component isomorphic to \( C_r \).

Suppose \( l \geq 2 \) and the theorem holds for cycles of length \( 2^{l-1} k \). Suppose \( r = 2^l k \) where \( k \) is odd and \( x_1, x_2, x_3, \ldots, x_r, x_1 \) form a component isomorphic to \( C_r \) in \( UG^c(D) \). It follows that there are two cycles of length \( 2^{l-1} k \) in \( dom^c(D) \). Since \( dom^c(D) \) is isomorphic to \( UG^c(D) \), these two cycles must also be in \( UG^c(D) \). By Theorem 3.11, they are either connected components isomorphic to \( C_{2^{l-1} k} \) or contained within components isomorphic to \( K_m \), where \( m \geq 2^{l-1} k \). However, the first case contradicts the inductive hypothesis. The second case is contradicted by Lemma 3.7 as each set would be independent in \( UG(D) \) and so could not be in \( C_r \) in \( UG^c(D) \).

Finally, we discuss two basic results related to the structure of the underlying graph when \( UG^c(D) = P_n \). The actual construction of a path in \( dom^c(D) \) relies upon careful orientation of specific edges in \( UG(D) \). Thus that portion
of the characterization is in Section 4, where the complete characterization of the digraph $D$, with $UG^c(D)$ connected, is developed.

Lemma 3.4 describes the odd length cycle that is created in $dom^c(D)$ when $UG^c(D)$ is an odd cycle. Here, we make a similar observation for the structure $UG^c(D) = P_n$. Unlike the case of the cycle, the path can have an odd or even number of vertices.

Lemma 3.17. If $UG^c(D) = P_n = x_1, x_2, \ldots, x_n$ for $n \geq 3$, then

1. if $n$ is odd, $x_1, x_3, \ldots, x_n$ and $x_2, x_4, \ldots, x_{n-1}$ are paths in $dom^c(D)$, and
2. if $n$ is even, $x_1, x_3, \ldots, x_{n-1}$ and $x_2, x_4, \ldots, x_n$ are paths in $dom^c(D)$.

Proof. Vertices $x_{i-1}$ and $x_{i+1}, i = 2, \ldots n - 1$, are not adjacent to vertex $x_i$ in $UG(D)$, so cannot dominate $x_i$. Thus $\{x_{i-1}, x_{i+1}\}$ is not an edge of $dom(D)$, but is an edge of $dom^c(D)$. This implies that $x_1, x_3, \ldots, x_n$ and $x_2, x_4, \ldots, x_{n-1}$ are paths in $dom^c(D)$ when $n$ is odd, while $x_1, x_3, \ldots, x_{n-1}$ and $x_2, x_4, \ldots, x_n$ are paths in $dom^c(D)$ when $n$ is even.

4. Characterization of $D$ where $UG^c(D)$ is Connected

Now that we know what each component of $UG^c(D)$ must be, we focus our attention on the case where $UG^c(D)$ is a single component.

What exact form do the digraphs take where $UG(D)$ is isomorphic to $dom(D)$ and $UG^c(D)$ is connected? To answer this question, we first introduce a simple result that links the degree of vertices in an underlying graph of any digraph to the existence of a $K_3$ in $dom^c(D)$. Although similar to the results in Section 2 regarding $K_3$ in $UG^c(D)$ and $dom^c(D)$, it is not identical. Here, the use of an orientation of an existing edge of $UG(D)$ does not translate into an adjacency issue in $UG^c(D)$. Thus, we approach the $K_3$ in $dom^c(D)$ through $dom(D)$.

Lemma 4.1. Let $D$ be a digraph on $n$ vertices, and $(u, v)$ be an arc in $D$ where $(v, u)$ is not an arc in $D$ and $\deg(u) = k$ in $UG(D)$. If $k < n - 2$, then $K_3$ is a subgraph of $dom^c(D)$.

Proof. Suppose $\deg(u) < n - 2$ in $UG(D)$. This implies that there are two vertices $x_1, x_2$ that are not adjacent to $u$ in $UG(D)$. Since in $D$ there is no
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arc from \(v\) to \(u\), then \(v, x_1, x_2\) do not dominate \(u\), so form an independent set in \(\text{dom}(D)\). Thus, they create a copy of \(K_3\) in \(\text{dom}^c(D)\).

Using the preceding result, we introduce the following two lemmas, which examine the existence of arcs that are not in a 2-cycle in some biorientation \(D\) of \(C_n^c\) or of \(P_n^c\) where \(UG(D) \cong \text{dom}(D)\).

**Lemma 4.2.** Let \(D\) be a directed graph on \(n \geq 5\) vertices, where \(n\) is odd and \(UG(D) = C_n^c\). Then, \(UG(D) \cong \text{dom}(D)\) if and only if \(D\) is symmetric.

**Proof.** Let \(D\) be a directed graph with \(UG^c(D) = C_n\), \(n \geq 5\). As \(UG^c(D) = C_n\), all degrees of the vertices of \(UG(D)\) are \(n - 3\). Suppose that \(UG(D) \cong \text{dom}(D)\). Then \(\text{dom}^c(D)\) contains no \(K_3\), and by Lemma 4.1, if \((u, v)\) is an arc in \(D\), then \((v, u)\) must also be an arc in \(D\). That is, \(D\) must be symmetric. On the other hand, if \(D\) is symmetric, we know by Theorem 2.2 that \(UG(D) \cong \text{dom}(D)\).

**Lemma 4.3.** Let \(D\) be a directed graph on \(n \geq 3\) vertices and \(UG^c(D) = P_n = x_1, \ldots, x_n\). Then, \(\text{dom}^c(D) \cong P_n\) if and only if every arc of \(D\) is in a two-cycle except

1. if \(n\) is odd, exactly one of the following sets of arcs are in \(D\) but not in a two-cycle:
   - (a) \((x_1, x_n)\),
   - (b) \((x_n, x_1)\),
   - (c) \((x_1, x_n)\) and \((x_n, x_{n-3})\), or
   - (d) \((x_n, x_1)\) and \((x_1, x_4)\),

2. if \(n\) is even, exactly one of the following sets of arcs are in \(D\) but not in a two-cycle:
   - (a) \((x_1, x_{n-1})\),
   - (b) \((x_n, x_2)\),
   - (c) \((x_1, x_{n-1})\) and \((x_n, x_2)\),
   - (d) \((x_n, x_2)\) and \((x_1, x_4)\), or
   - (e) \((x_1, x_{n-1})\) and \((x_n, x_{n-3})\).

**Proof.** Let \(D\) be a directed graph such that \(UG^c(D) = P_n = x_1, \ldots, x_n\) for \(n \geq 3\). Lemma 3.17 shows that \(\text{dom}^c(D)\) contains the edges \(\{x_1, x_3\}, \ldots, \{x_{n-2}, x_n\}, \{x_2, x_4\}, \ldots, \{x_{n-3}, x_{n-1}\}\) if \(n\) is odd, or the edges \(\{x_1, x_3\}, \ldots, \{x_{n-2}, x_n\}, \{x_2, x_4\}, \ldots, \{x_{n-3}, x_{n-1}\}\) if \(n\) is even.
\{x_{n-3}, x_{n-1}\}, \{x_2, x_4\}, \ldots, \{x_{n-2}, x_n\} if \(n\) is even. Thus \(\text{dom}^c(D)\) contains at least \(n - 2\) edges.

\((\Rightarrow)\) Suppose \(\text{dom}^c(D) \cong P_n\). Thus \(UG^c(D) \cong \text{dom}^c(D)\) and \(UG(D) \cong \text{dom}(D)\). By Theorem 2.2, since \(UG^c(D) \cong P_n\), \(D\) cannot be symmetric. Thus, at least one arc must not be in a two-cycle of \(D\).

Since \(\text{dom}^c(D) \cong P_n\), \(\text{dom}^c(D)\) contains no subgraph isomorphic to \(K_3\). Thus, for any arc \((u, v)\) in \(D\) where \((v, u)\) is not in \(D\), Lemma 4.1 states \(\text{deg}(u) \geq n - 2\) in \(UG^c(D)\). Vertices \(x_1\) and \(x_n\) are the only vertices that meet this criterion. This indicates that if \((u, v)\) is an arc in \(D\) but \((v, u)\) is not, \(u = x_1\) or \(u = x_n\).

Suppose that \((x_1, x_i)\) is an arc, but \((x_i, x_1)\) is not. Then, \(\{x_2, x_i\}\) is not an edge in \(\text{dom}(D)\), since neither dominates \(x_1\) in \(D\). This ensures that \(\{x_2, x_i\}\) is an edge in \(\text{dom}^c(D)\). If \(i = 3\) or \(5 \leq i \leq n - 2\), then \(x_i\) will be adjacent to three vertices, \(x_{i-2}, x_{i+2}\) and \(x_2\) in \(\text{dom}^c(D)\), but \(P_n\) has no vertices of degree three. Also, by the structure of \(UG^c(D)\), there are no arcs between \(x_1\) and \(x_2\). Consequently, there are three possibilities to consider: \(i = 4, i = n - 1,\) or \(i = n\). Likewise if \((x_n, x_i)\) is an arc but \((x_i, x_n)\) is not, then \(i = 1, i = 2,\) or \(i = n - 3\).

Suppose that both \((x_1, x_i)\) and \((x_1, x_j)\) are arcs in \(D\), with \(i \neq j\), but neither \((x_i, x_1)\) nor \((x_j, x_1)\) is an arc in \(D\). Then, \(x_1, x_j,\) and \(x_2\) all fail to dominate \(x_1\). Consequently, \(x_1, x_j,\) and \(x_2\) form a \(K_3\) in \(\text{dom}^c(D)\), which is impossible if \(\text{dom}^c(D)\) is isomorphic to a path. Thus at most one of \((x_4, x_1), (x_{n-1}, x_1), (x_n, x_1)\) is missing from \(D\). Similarly, at most one of \((x_1, x_n), (x_2, x_n), (x_{n-3}, x_n)\) is missing from \(D\). All other arcs must be in two-cycles.

If \((x_1, x_i)\) is an arc in \(D\), but \((x_i, x_1)\) is not, \(x_1\) and \(x_2\) fail to dominate \(x_1\), so \(\{x_2, x_i\}\) must be an edge in \(\text{dom}^c(D)\). If \(i = 4,\) by Lemma 3.17, \(\{x_2, x_i\}\) is already an edge in \(\text{dom}^c(D)\). If \(n\) is odd, \(\{x_2, x_{n-1}\}\) cannot be an edge in \(\text{dom}^c(D)\), since \(x_2, \ldots, x_{n-1}\) would form a cycle, which is impossible. Thus if \(n\) is odd, \(i \neq n - 1\). If \(n\) is even, \(\{x_2, x_n\}\) cannot be an edge in \(\text{dom}^c(D)\), otherwise \(x_2, \ldots, x_n\) would form a cycle. Thus if \(n\) is even, \(i \neq n\). Similarly, if we assume \((x_n, x_i)\) is an arc but \((x_i, x_n)\) is not, \(\{x_i, x_{n-1}\}\) is an edge in \(\text{dom}^c(D)\), and if \(n\) is odd, \(i \neq 2\), and if \(n\) is even \(i \neq 1\).

Since we need at least one additional edge in \(\text{dom}^c(D)\) to form a path, at least one of \((x_n, x_1), (x_{n-1}, x_1), (x_1, x_n), (x_2, x_n)\) must be missing from \(D\). Finally, notice that, since \(\{x_1, x_n\}\) is an edge in \(UG(D)\), at least one of \((x_1, x_n), (x_n, x_1)\) is an arc in \(D\).

Consequently, every arc in \(D\) will be in a two cycle, except for one of the following cases: If \(n\) is odd, either \((x_1, x_n)\) is not in \(D\), \((x_1, x_n)\) and \((x_4, x_1)\)
are not in $D$, $(x_n, x_1)$ is not in $D$, or $(x_n, x_1)$ and $(x_{n-3}, x_n)$ are not in $D$. If $n$ is even, either $(x_2, x_n)$ is not in $D$, $(x_2, x_n)$ and $(x_4, x_1)$ are not in $D$, $(x_{n-1}, x_1)$ is not in $D$, $(x_{n-1}, x_1)$ and $(x_{n-3}, x_n)$ is not in $D$, or $(x_{n-1}, x_1)$ and $(x_2, x_n)$ is not in $D$.

$(\Leftarrow)$ Suppose $D$ has one of the patterns in the previous paragraph. If $n$ is odd, and $(x_1, x_n)$ is an arc in $D$, then $(x_2, x_n)$ will already be in $\text{dom}^c(D)$ and thus $x_1, x_3, \ldots, x_n, x_2, x_4, \ldots, x_{n-1}$ is a path in $\text{dom}^c(D)$. Similarly, if $(x_n, x_1)$ is an arc in $D$, then $(x_1, x_n)$ is not, then $\{x_1, x_{n-1}\}$ is an edge in $\text{dom}^c(D)$. Thus $\text{dom}^c(D)$ contains the path $x_2, x_4, \ldots, x_{n-1}, x_1, \ldots, x_n$. If $n$ is even, since either $(x_1, x_{n-1})$ is an arc but $(x_n, x_1)$ is not, or $(x_n, x_2)$ is an arc but $(x_2, x_n)$ is not an arc in $D$, $(x_2, x_{n-1})$ is an edge in $\text{dom}^c(D)$. Thus $\{x_1, x_3, \ldots, x_{n-1}, x_2, \ldots, x_n\}$ is a path in $\text{dom}^c(D)$.

Now consider any pair of vertices $x_i, x_j$, not in a path described above, with $i < j$. Note that $j \neq i + 2$. By the construction of $D$, every vertex $x_i$ has an arc to every vertex $x_k$ with the following possible exceptions: $k = i, i - 1, i + 1, 1, n$. Suppose that there is an edge between $x_i$ and $x_j$ in $\text{dom}^c(D)$. There must be some $k \neq i, j$ so that neither $(x_i, x_k)$ is an arc, nor $(x_j, x_k)$ is an arc. Since $i + 1 \neq j - 1$, this means $k = 1$ or $k = n$. Suppose $k = 1$. By the way $D$ is constructed, there are only a limited number of possibilities. $i$ must equal 2. If $n$ is odd, $j$ might equal $n$, in which case $(x_2, x_n)$ will already be in $\text{dom}^c(D)$. If $n$ is even, $j$ might equal $n - 1$, in which case $(x_2, x_{n-1})$ is already in $\text{dom}^c(D)$. In either case no additional edges are in $\text{dom}^c(D)$. Similar arguments suffice for $k = n$. In either case $\text{dom}^c(D) = P_n$.

![Figure 4. Example biorientations of the complement of paths.](image)

It is now possible to completely characterize all digraphs with isomorphic underlying graphs and domination graphs where $\text{UG}^c(D)$ is connected.
Theorem 4.4. For any digraph $D$, $UG(D) \cong \text{dom}(D)$ and $UG^c(D)$ is connected if and only if

1. $D$ is a digraph of $k$ isolated vertices other than $k = 2$, or
2. $D$ is a complete biorientation of $C_n^c$ where $n$ is odd, or
3. $D$ is the biorientation of $P_n^c$ where $P_n = x_1, \ldots, x_n$ with $n \geq 3$, and
   (a) if $n$ is odd, exactly one of the following sets of arcs are in $D$ but not in a two-cycle:
      (i) $(x_1, x_n)$,
      (ii) $(x_n, x_1)$,
      (iii) $(x_1, x_n)$ and $(x_n, x_{n-3})$, or
      (iv) $(x_n, x_1)$ and $(x_1, x_4)$, and
   (b) if $n$ is even, exactly one of the following sets of arcs are in $D$ but not in a two-cycle:
      (i) $(x_1, x_{n-1})$,
      (ii) $(x_n, x_2)$,
      (iii) $(x_1, x_{n-1})$ and $(x_n, x_2)$,
      (iv) $(x_n, x_2)$ and $(x_1, x_4)$, or
      (v) $(x_1, x_{n-1})$ and $(x_n, x_{n-3})$.
   (c) for all other arcs $(x, y)$ of $D$, $(y, x)$ is also an arc.

Proof. Recall that Theorem 3.11 states that $UG^c(D)$ must be a complete graph, a cycle, or a path, so $D$ must be an orientation of the complement of such graphs. We know from Theorem 2.2 that $D$ exists with $UG^c(D) = K_n$ for $n \neq 2$ or $C_n$, where $n$ is odd. Lemma 4.3 provides constructions when $UG^c(D) \cong P_n$, $n \geq 3$.

If $UG^c(D)$ is isomorphic to $K_n$ then $D$ has no arcs so there is no choice of orientation. If $n = 2$, then $\text{dom}(D) = K_2 \not\cong UG(D)$. In all other cases $\text{dom}(D)$ will be isomorphic to $k$ isolated vertices.

Suppose $UG^c(D)$ is an odd length cycle. If $n = 3$ then $D$ is isomorphic to 3 isolated vertices listed in the previous case. If $n \geq 5$, Lemma 4.2 states that $D$ must be a complete biorientation of $UG(D)$, the complement of $C_n$. It follows from Lemma 3.14 that if $UG^c(D)$ is an even length cycle on more than 4 vertices then $\text{dom}^c(D)$ will contain two smaller cycles which will result in a disconnected graph. Finally, let $x_1, x_2, x_3, x_4, x_1$ be a 4 cycle in $UG^c(D)$, so $UG(D)$ has two edges $\{x_1, x_3\}$ and $\{x_2, x_4\}$. There are only
three non-isomorphic orientations of two edges. Each of these orientations results in $dom^c(D) \not\sim C_4$.

The final case, where $UG^c(D)$ is isomorphic to a path is fully described in Lemma 4.3.

References


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