NEW SUFFICIENT CONDITIONS FOR HAMILTONIAN AND PANCYCLIC GRAPHS

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Abstract

For a graph $G$ of order $n$ we consider the unique partition of its vertex set $V(G) = A \cup B$ with $A = \{v \in V(G) : d(v) \geq n/2\}$ and $B = \{v \in V(G) : d(v) < n/2\}$. Imposing conditions on the vertices of the set $B$ we obtain new sufficient conditions for hamiltonian and pancyclic graphs.

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1. Introduction

We use [4] for terminology and notation not defined here and consider finite and simple graphs only.

A graph of order $n$ is called hamiltonian if it contains a cycle of length $n$ and is called pancyclic if it contains cycles of all lengths from 3 to $n$.

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Let $\omega(G)$ denote the number of components of a graph $G$. A graph $G$ is called 1-tough if, for every nonempty proper subset $S$ of $V(G)$, we have $\omega(G - S) \leq |S|$.

Various sufficient conditions for a graph to be hamiltonian have been given in terms of vertex degrees. Recall some of them.

**Theorem 1** (Dirac [5]). Let $G$ be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then $G$ is hamiltonian.

**Theorem 2** (Ore [9]). Let $G$ be a graph of order $n \geq 3$. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u, v \in V(G)$, then $G$ is hamiltonian.

In [7] Theorem 2 was extended as follows.

**Theorem 3** (Flandrin, Li, Marczyk, Woźniak [7]). Let $G = (V, E)$ be a 2-connected graph on $n$ vertices with minimum degree $\delta$. If $uv \in E(G)$ for every pair of vertices $u, v \in V(G)$ with $d(u) = \delta$ and $d(v) < n/2$, then $G$ is hamiltonian.

With respect to its vertex degrees, the vertex set of every graph $G$ has a unique partition $V(G) = A \cup B$ with $A = \{v \in V(G) : d(v) \geq n/2\}$ and $B = \{v \in V(G) : d(v) < n/2\}$. In terms of $A$ and $B$ we make the following observations:

- If a graph $G$ satisfies Dirac’s condition then $B = \emptyset$.
- If a graph $G$ satisfies Ore’s condition, then $G[B]$ is complete and $|B| \leq \delta + 1$.
- If a graph $G$ satisfies the condition of Theorem 3, then $G[B]$ is connected, $G[u \in B : d(u) = \delta]$ is complete and $|B| \leq \delta + 1$.

## 2. Results

We define three classes of graphs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ as follows.

Let $\mathcal{G}_1$ be the class of all 2-connected graphs $G$ such that $uv \in E(G)$ for every pair of vertices $u, v \in B$ with $d(u) = \delta(G)$.

Let $\mathcal{G}_2$ be the class of all 2-connected graphs $G$ such that there exists a vertex $u \in B$ with $d(u) = \delta(G)$ and $uv \in E(G)$ for all vertices $v \in B - \{u\}$.

Let $\mathcal{G}_3$ be the class of all 2-connected graphs $G$ such that $|B| \leq \delta(G) + 1$ and $\Delta(G[B]) \geq \min\{2, |B| - 1\}$.
Figure 1. Graph $F_{n,\delta}$

For all $n, \delta$ with $2 \leq \delta \leq \frac{n-1}{2}$ define $F_{n,\delta}$ as a graph of order $n$, minimum degree $\delta$ and vertex set $V(F_{n,\delta}) = \{u_0, u_1, \ldots, u_\delta, w_1, \ldots, w_{n-\delta-1}\}$ such that $d(u_0) = \delta$, $N(u_0) = \{u_1, \ldots, u_\delta\}$, vertices $u_1, \ldots, u_\delta$ are independent, vertices $w_1, \ldots, w_{n-\delta-1}$ induce a clique and $u_iw_j \in E(G)$ for all $1 \leq i \leq \delta$ and $1 \leq j \leq \delta - 1$. Now, for $S = \{u_0, w_1, \ldots, w_{\delta-1}\}$ we have

$$\omega(F_{n,\delta} - S) = \delta + 1 > \delta = |S|.$$ 

Hence, $F_{n,\delta}$ is not 1-tough and therefore not hamiltonian.

For all $n, \delta$ with $2 \leq \delta \leq \frac{n-1}{2}$ define $H_{n,\delta}$ as a supergraph of $F_{n,\delta}$ such that $V(H_{n,\delta}) = V(F_{n,\delta})$ and $E(H_{n,\delta}) = E(F_{n,\delta}) \cup \{u_0w_i : 1 \leq i \leq n - \delta - 1\}$. Hence, $H_{n,\delta}$ is not 1-tough and therefore not hamiltonian, too.

Theorem 3 can be now restated as follows.

**Theorem 3 (restated).**

If $G \in \mathcal{G}_1$, then $G$ is hamiltonian.

Using closure operations we obtain the following extension of Theorem 3.

**Theorem 4.** If $G \in \mathcal{G}_2$, then $G$ is hamiltonian or $G \subset F_{n,\delta}$.

The proof of the above theorem is given in Section 3. It provides a further extension which can be formulated as follows.

**Theorem 5.** If $G \in \mathcal{G}_3$, then $G$ is hamiltonian or $G \subset H_{n,\delta}$.

Since both $F_{n,\delta}$ and $H_{n,\delta}$ are not 1-tough, we obtain the following corollary.
Corollary 6. If $G \in \mathcal{G}_3$ is 1-tough, then $G$ is hamiltonian.

Bondy suggested the interesting ”meta-conjecture” in [2] that almost any nontrivial condition on graphs which implies that the graph is hamiltonian also implies that the graph is pancyclic (there may be a family of exceptional graphs). He proved the following result concerning Ore’s condition.

Theorem 7 ([2]). Let $G$ be a graph of order $n \geq 3$. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u, v \in V(G)$, then $G$ is pancyclic or isomorphic to the complete bipartite graph $K_{n/2, n/2}$.

In [7] it was shown that Theorem 7 can be extended as follows.

Theorem 8. If $G \in \mathcal{G}_1$, then $G$ is pancyclic or $G \cong K_{n/2, n/2}$.

Theorem 3 extends the following result of Jin, Liu and Wang [8].

Corollary 9 ([8]). Let $G$ be a 2-connected graph of order $n \geq 3$. If $d(u) + d(v) \geq n + \delta$ for every pair of nonadjacent vertices $u, v \in V(G)$, then $G$ is pancyclic or $G \cong K_{n/2, n/2}$.

Concerning pancyclicity we will prove the following theorems.

Theorem 10. If $G \in \mathcal{G}_2$, then $G$ is pancyclic or $G \cong K_{n/2, n/2}$ or $G \subset F_{n, \delta}$.

Theorem 11. If $G \in \mathcal{G}_3$, then $G$ is pancyclic or bipartite or $G \subset H_{n, \delta}$.

3. Proofs

3.1. Hamiltonicity

The closure concept of Bondy and Chvátal [3] is based on the following result of Ore [9].

Theorem 12 (Ore [9]). Let $G$ be a graph on $n$ vertices such that the edge $e = uv$ does not belong to $E(G)$ and $d(u) + d(v) \geq n$. Then, the graph $G$ is hamiltonian if and only if the graph $G + e$ is hamiltonian.

By successively joining pairs of nonadjacent vertices having degree sum at least $n$ as long as this is possible (in the new graph(s)), the unique so called $n$-closure $cl_n(G)$ is obtained. Using Theorem 12 it is easy to prove the following result.
Theorem 13 (Bondy and Chvátal [3]). Let $G$ be a graph of order $n \geq 3$. Then $G$ is hamiltonian if and only if $cl_n(G)$ is hamiltonian.

Corollary 14 (Bondy and Chvátal [3]). Let $G$ be a graph of order $n \geq 3$. If $cl_n(G)$ is complete ($cl_n(G) = K_n$), then $G$ is hamiltonian.

Ainouche and Christofides [1] established the following generalization of Theorem 12.

Theorem 15 (Ainouche and Christofides [1]). Let $G$ be a 2-connected graph on $n$ vertices such that the edge $e = uv$ does not belong to $E(G)$. Let $T = T(u, v) = \{w \in V(G) \setminus (N[u] \cup N[v])\}$ and let $t = |T|$. Suppose that
\[(*) \quad d(w) \geq t + 2 \quad \text{for all vertices of } T.\]

Then, the graph $G$ is hamiltonian if and only if the graph $G + e$ is hamiltonian.

In [1] the corresponding (unique) closure of $G$ is called the 0-dual closure $cl^*(G)$. Since Theorem 15 is more general than Theorem 12 (cf. [1]), $G \subseteq cl_n(G) \subseteq cl^*(G)$. The counterpart of Corollary 14 is

Corollary 16 (Ainouche and Christofides [1]). Let $G$ be a 2-connected graph of order $n$. If $cl^*(G)$ is complete ($cl^*(G) = K_n$), then $G$ is hamiltonian.

Proof of Theorem 4. Observe first that if $\delta(G) \geq \frac{n}{2}$ then $G$ is hamiltonian by Dirac’s theorem. So, assume that $B \neq \emptyset$.

Step 0. Applying the Bondy-Chvátal closure to the set $A$ we get the graph $G_0$ with the set $A$ complete.

Step 1. By using $cl^*$ we are able to add to $G_0$ all edges connecting the vertex $u$ with the set $A$. Indeed, it suffices to verify the hypothesis of Theorem 15. Suppose there exists a vertex $x \in A$ such that $ux \notin E$. Since $u$ is adjacent to all vertices of $B$ and $x$ is adjacent (in $G_0$) to all vertices of $A$, we have $T_{G_0}(u, x) = \emptyset$. Denote the graph obtained in this step by $G_1$.

Step 2. Let $x \in B, x \neq u$. We put $a(x) = a_G(x) = |N_G(x) \cap A|$. Denote by $B'$ the vertices of $B$ with $a(x) < \delta - 1$. Consider now a vertex $x \in B'$ and
a vertex \( y \in A \) such that \( xy \notin E \). Then \( x \) has at least one neighbour in \( B \) different from \( u \). This implies \( |T(x, y)| \leq \delta - 2 \). Hence, the condition (\( \ast \)) of Theorem 15 is satisfied. This means that we can add all edges between \( B' \) and \( A \). We denote the graph obtained in this step by \( G_2 \).

**Step 3.** Denote by \( B_1 \) the vertices of \( B \) different from \( u \) that are joined to all vertices of \( A \) in \( G_2 \). Note that \( B' \subset B_1 \). We put \( B_2 = B - (B_1 \cup \{ u \}) \).

Let \( \xi = |B_1| \) and \( \eta = |B_2| \). We have \( 1 + \xi + \eta = |B| \leq \delta + 1 \). Consider now a vertex \( x \in B_2 \) and \( y \in A \) such that \( xy \notin E \). By Step 2, \( a_{G_2}(x) \geq \delta - 1 \).

Since the vertices of \( A \) as well as the vertices of \( B_1 \) and the vertex \( u \) are the neighbours of \( y \) we get \( |T(x, y)| \leq \eta - 1 \). So, \( t + 2 \leq \eta + 1 \). If the condition (\( \ast \)) of Theorem 15 is not fulfilled then \( \delta \leq \eta \). This implies in particular that \( \xi = 0 \) and \( \delta = \eta \). Moreover, \( x \) has no neighbour in \( B \) other than \( u \), for otherwise \( |T(x, y)| \leq \eta - 2 \) and (\( \ast \)) would be satisfied. Observe that either

(a) the above statements concern all vertices of \( B_2 \) (see Step 5), or

(b) we can add all edges between \( B_2 \) and \( A \).

In the later case we can continue the closure operation (see Step 4 below).

**Step 4.** Denote by \( G_3 \) the graph obtained in Step 3b. Let \( x, y \) be two vertices of \( B \) such that \( xy \notin E \). Then at most \( \delta - 2 \) vertices of \( B \) belong to \( T(x, y) \) and we can finish the closure operation with the conclusion that \( \text{cl}^*(G) = K_n \).

**Step 5.** Suppose now that no edge can be added in Step 3. Then \( B \) consists of the vertex \( u \) and its \( \delta \) neighbours, say \( u_1, u_2, \ldots, u_\delta \), forming an independent set. This implies that each of the vertices \( u_1, u_2, \ldots, u_\delta \) sends at least \( \delta - 1 \) edges to \( A \). Suppose, that there exists a vertex \( x \in A \) such that \( u_ix \notin E \) and \( u_jx \in E \) for some \( j \neq i \). Then \( |T(u_i, x)| \leq \delta - 2 \) and the edge \( u_ix \) can be added to \( G_2 \). Denote by \( G_5 \) the graph obtained from \( G_2 \) by adding all edges as above. We may conclude that in \( G_5 \) all vertices \( u_1, u_2, \ldots, u_\delta \) have the same neighbourhood. It is now easy to see that only in the case where this neighbourhood contains exactly \( \delta - 1 \) vertices of \( A \) the graph \( G_5 \) is not hamiltonian. Observe that in this case \( G_5 \subset F_{n,\delta} \).

**Proof of Theorem 5.** As in the previous proof observe first that if \( \delta(G) \geq \frac{n}{2} \) then \( G \) is hamiltonian by Dirac’s theorem. So, assume that \( B \neq \emptyset \). Applying the Bondy-Chvátal closure to the set \( A \) we get the graph \( G_0 \) with the set \( A \) complete. It is easy to verify the hamiltonicity of the
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3.2. Pancyclicity

For the proof of Theorem 11 we will apply the following three theorems.

**Theorem 17** (Faudree, Häggkvist, Schelp [6]). *Every hamiltonian graph of order n and size \( e(G) > \frac{(n-1)^2}{4} + 1 \) is pancyclic or bipartite.*

**Lemma 18** (Bondy [2]). *Let \( G \) be a hamiltonian graph of order \( n \) with a Hamilton cycle \( v_1v_2...v_nv_1 \) such that \( d(v_1) + d(v_n) \geq n + 1 \). Then \( G \) is pancyclic.*

**Theorem 19** (Schmeichel-Hakimi [10]). *If \( G \) is a hamiltonian graph of order \( n \geq 3 \) with a Hamilton cycle \( v_1v_2...v_nv_1 \) such that \( d(v_1) + d(v_n) \geq n \), then \( G \) is either

- pancyclic,
- bipartite, or
- missing only an \((n - 1)\)-cycle.

Moreover, in the last case we have \( d(v_{n-2}), d(v_{n-1}), d(v_2), d(v_3) < n/2 \).*

**Remark.** Actually, the Schmeichel-Hakimi result gives some more information about the possible adjacency structure near the vertices \( v_1 \) and \( v_n \), but the above version is sufficient for our proof.
**Proof of Theorem 11.** If $\delta \geq n/2$, then $G$ is pancyclic or $G \cong K_{\frac{n}{2}, \frac{n}{2}}$ by Theorem 7. Hence we may assume that $2 \leq \delta \leq \frac{n-1}{2}$. If $G \in H_{n,\delta}$, then $G$ is not hamiltonian and thus not pancyclic. Hence we may further assume that $G$ is hamiltonian.

If $\delta = \frac{n-1}{2}$, then

$$e(G) \geq \frac{1}{2} \left(\frac{n+1}{2} \cdot \frac{n-1}{2} + \frac{n-1}{2} \cdot \frac{n+1}{2}\right) = \frac{n^2 - 1}{4} > \frac{(n-1)^2}{4} + 1$$

for all $n \geq 5$. Thus $G$ is pancyclic or bipartite by Theorem 17 (since $n$ is odd, $G$ cannot be bipartite).

If $\delta = \frac{n-3}{2}$, then

$$e(G) \geq \frac{1}{2} \left(\frac{n}{2} \cdot \frac{n-2}{2} + \frac{n}{2} \cdot \frac{n}{2}\right) = \frac{n^2 - n}{4} > \frac{(n-1)^2}{4} + 1$$

for all $n \geq 6$. Thus $G$ is pancyclic or bipartite by Theorem 17. In the later case we conclude that $G \cong K_{\frac{n}{2}, \frac{n}{2}} - \frac{n}{4}K_2$.

If $2 \leq \delta \leq \frac{n-3}{2}$, then $|A| \geq \frac{n+1}{2} > |B|$, since $|B| \leq \delta + 1 \leq \frac{n-1}{2}$. In this case the third alternative of Theorem 19 cannot occur since a simple counting argument gives $|A| \leq |B|$, a contradiction. Hence $G$ is pancyclic or bipartite by Theorem 19.

**Proof of Theorem 10.** Since $G_2 \subset G_3$ we can apply Theorem 11. If $G \in H_{n,\delta}$ then we conclude that $G \in F_{n,\delta}$, since there is a vertex $u \in B$ with $d(u) = \delta$. Suppose $G \notin F_{n,\delta}$. Then $G$ is hamiltonian by Theorem 4. Thus, if $G$ is bipartite, then $G$ is balanced bipartite with partite sets $V_1$ and $V_2$. Suppose $u \in V_1$ for a vertex $u$ with $d(u) = \delta$ and $wv \in E(G)$ for all vertices $v \in B - \{u\}$. Since $\delta < n/2$, there exists a vertex $w \in V_2$ with $w \notin N(u)$. But then $d(w) \leq \frac{n}{2} - 1 < \frac{n}{2}$, a contradiction. Thus $G$ cannot be bipartite. Therefore, by Theorem 11, $G$ is pancyclic.

### 4. Concluding Remarks

Our results presented in Section 2 all imply that $|B| \leq \delta + 1$ for the considered graphs. Thus it is a natural question to study hamiltonicity (and pancyclicity) of graphs with $|B| \leq \delta + k$ for some positive integer $k \geq 2$. 
For all $n, \delta$ and $k$ with $2 \leq \delta \leq \frac{n-1}{2}$ and $1 \leq k \leq \delta - 1$ define $I_{n,\delta,k}$ as a graph of order $n$, minimum degree $\delta$ and vertex set

$$V(I_{n,\delta,k}) = \{u_1, \ldots, u_k, v_1, \ldots, v_\delta, w_1, \ldots, w_{n-\delta-k}\}$$

such that $d(u_i) = n-1$ for $1 \leq i \leq k$, the vertices $\{v_1, \ldots, v_\delta\}$ are independent, $G'[\{w_1, \ldots, w_{n-\delta-k}\}]$ is complete and $v_i w_j \in E(G)$ for all $1 \leq i \leq \delta$ and $1 \leq j \leq \delta - k$.

Now, for $S = \{u_1, \ldots, u_k, w_1, \ldots, w_{\delta-k}\}$ we have

$$\omega(I_{n,\delta,k} - S) = \delta + 1 > \delta = |S|.$$ 

Hence, $I_{n,\delta,k}$ is not 1-tough and thus not hamiltonian. Note that $I_{n,\delta,1} = H_{n,\delta}$.

Following the proof of Theorem 5 we have obtained the following theorem.

**Theorem 20.** Let $G$ be a 2-connected graph of order $n$. If for some $k$ with $1 \leq k \leq \delta - 1$

(i) $G[B]$ is complete for $|B| \leq k + 1$ or

(ii) there are at least $k$ vertices of degree at least $k + 1$ in $B$ for $k + 2 \leq |B| \leq \delta + k$,

then $G$ is hamiltonian or $\subset I_{n,\delta,k}$. 

References


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