ON DOMINATION IN GRAPHS

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Abstract

For a finite undirected graph $G$ on $n$ vertices two continuous optimization problems taken over the $n$-dimensional cube are presented and it is proved that their optimum values equal the domination number $\gamma$ of $G$. An efficient approximation method is developed and known upper bounds on $\gamma$ are slightly improved.

Keywords: graph, domination.

2000 Mathematics Subject Classification: 05C69.

1. Introduction and Results

For terminology and notation not defined here we refer to [3]. Let $V = V(G) = \{1, \ldots, n\}$ be the vertex set of an undirected graph $G$, and for $i \in V$, $N(i)$ be the neighbourhood of $i$ in $G$, $N_2(i) = \{k \in V \mid k \in \bigcup_{j \in N(i)} N(j) \setminus \{i\} \cup N(i)\}$, $d_i = |N(i)|$, $t_i = |N_2(i)|$, $\delta = \min_{i \in V} d_i$, and $\Delta = \max_{i \in V} d_i$.

A set $D \subseteq V(G)$ is a dominating set of $G$ if $(\{i\} \cup N(i)) \cap D \neq \emptyset$ for every $i \in V$. The minimum cardinality of a dominating set of $G$ is the domination number $\gamma = \min \{|D| \mid D \text{ is a dominating set of } G\}$. A dominating set $D$ is minimal if $D$ is a dominating set of $G$ and there is no dominating set of $G$ with cardinality smaller than $|D|$. A set $D$ is a maximal dominating set if $D$ is a dominating set of $G$ and there is no dominating set of $G$ with cardinality larger than $|D|$.
number $\gamma$ of $G$. In [7] $\gamma = \min_{x_1, \ldots, x_n \in [0,1]} \sum_{i \in V} (x_i + (1-x_i) \prod_{j \in N(i)} (1-x_j))$ was proved. With $x_1 = \ldots = x_n = x$ we have $\gamma \leq (x + (1-x)^{\delta+1})n \leq (x + e^{-\delta x})n$ for every $x \in [0,1]$. Minimizing $x + (1-x)^{\delta+1}$ and $x + e^{-\delta x}$, the well-known inequalities $\gamma \leq (1 - \frac{1}{(\delta+1)^2} + \frac{1}{(\delta+1)^{\frac{3}{2}}})n \leq \frac{1+\ln(\delta+1)}{\delta+1}n$ (see [4, 8]) follow. Obviously, it is easily checked whether $\gamma = 1$ or not. Thus, we will assume $G \in \Gamma$ in the sequel, where $\Gamma$ is the set of graphs $G$ such that each component of $G$ has domination number greater than 1. Without mentioning in each case, we will use $d_i, t_i \geq 1$ for $i = 1, \ldots, n$ if $G \in \Gamma$. For $x_1, \ldots, x_n \in [0,1]$ let

$$f_G(x_1, \ldots, x_n) = \sum_{i \in V} \left( x_i \left( 1 - \prod_{j \in N(i)} x_j \right) \left( 1 - \prod_{k \in N_2(i)} x_k \right) \right) + (1-x_i) \prod_{j \in N(i)} (1-x_j)$$

$$g_G(x_1, \ldots, x_n) = f_G(x_1, \ldots, x_n) - \sum_{i \in V} \left( \frac{1}{1 + d_i} (1-x_i) \left( \prod_{j \in N(i)} (1-x_j) \right) \left( \prod_{k \in N_2(i)} (1-x_k) \right) \right).$$

**Theorem 1.** If $G \in \Gamma$ then

$$\gamma = \min_{x_1, \ldots, x_n \in [0,1]} f_G(x_1, \ldots, x_n) = \min_{x_1, \ldots, x_n \in [0,1]} g_G(x_1, \ldots, x_n)$$

$$\leq \min_{x \in [0,1]} \sum_{i \in V} \left( x \left( 1 - x^{d_i} (1-x^{t_i}) \right) + (1-x)^{d_i+1} \left( 1 - \frac{1}{1 + d_i} (1-x)^{t_i} \right) \right)$$

$$\leq \min_{x \in [0,1]} \left( x \left( 1 - x^{\Delta} (1-x) \right) + (1-x)^{\Delta+1} \left( 1 - \frac{1}{1 + \Delta} (1-x)^{\Delta(\Delta-1)} \right) \right)n.$$

Since DOMINATING SET is an NP-complete decision problem ([5]), it is difficult to solve the continuous optimization problem $\mathcal{P}$:

$$\min_{x_1, \ldots, x_n \in [0,1]} g_G(x_1, \ldots, x_n).$$

However, if $(x_1, \ldots, x_n)$ is the solution of any approximation method for $\mathcal{P}$, then (see Theorem 2) we can easily find a dominating set of $G$ of cardinality at most $g_G(x_1, \ldots, x_n)$. 


Theorem 2. Given a graph \( G \in \Gamma \) on \( V = \{1, \ldots, n\} \) with maximum degree \( \Delta_1 \), \( x_1, \ldots, x_n \in [0, 1] \), there is an \( O(\Delta^4n) \)-algorithm finding a dominating set \( D \) of \( G \) with \( |D| \leq g_G(x_1, \ldots, x_n) \).

2. Proofs

Proof of Theorem 1. For events \( A \) and \( B \) and for a random variable \( Z \) of an arbitrary random space, \( P(A), P(A|B) \), and \( E(Z) \) denote the probability of \( A \), the conditional probability of \( A \) given \( B \), and the expectation of \( Z \), respectively. Let \( \bar{A} \) be the complementary event of \( A \). We will use the well-known facts that \( P(B)P(A|B) = P(A \cap B) = P(B) - P(\bar{A} \cap B) = P(B)(1 - P(\bar{A}|B)) \) and \( E(|S'|) = \sum_{s \in S} E(s \in S') \) for a random subset \( S' \) of a given finite set \( S \). \( I \subseteq V \) is an independent set if \( N(i) \cap I = \emptyset \) for all \( i \in I \).

Consider fixed \( x_1, \ldots, x_n \in [0, 1] \). \( X \subseteq V \) is formed by random and independent choice of \( i \in V \), where \( P(i \in X) = x_i \). Let \( X' = \{ i \in X \mid N(i) \subseteq X \} \), \( X'' = \{ i \in X' \mid N(i) \cap (X \setminus X') \neq \emptyset \} \). \( Y = \{ i \in V \mid i \notin X \land N(i) \cap X = \emptyset \} \), \( Y' = \{ i \in Y \mid N(i) \cap Y \neq \emptyset \} \), and \( I \) be a maximum independent set of the subgraph of \( G \) induced by \( Y' \).

Lemma 3. \((X \setminus X'') \cup (Y \setminus I)\) is a dominating set of \( G \).

Proof. Obviously, \( X'' \subseteq X' \subseteq X \) and \( (X \setminus X') \subseteq (X \setminus X'') \). If \( i \in V \setminus (X \cup Y) \) then \( N(i) \cap (X \setminus X') \neq \emptyset \). If \( i \in X'' \) then again \( N(i) \cap (X \setminus X') \neq \emptyset \), and if \( i \in I \) then \( N(i) \cap (Y \setminus I) \neq \emptyset \).

Lemma 4. \( \gamma \leq E(|X|) - E(|X'')| + E(|Y|) - E(|I|) \).

Proof. Let \( \mathcal{D} \) be a random dominating set of \( G \). Because of the property of the expectation to be an average value we have \( \gamma \leq E(|D|) \). With Lemma 3 and linearity of the expectation, \( \gamma \leq E((|X \setminus X''| \cup |Y \setminus I|)) = E(|X| - |X''|) + E(|Y| - |I|) = E(|X|) - E(|X''|) + E(|Y|) - E(|I|) \) since \((X \setminus X'') \cap (Y \setminus I) = \emptyset \).

Lemma 5. \( E(|X|) = \sum_{i \in V} x_i, E(|X'') = \sum_{i \in V} x_i \left( \prod_{j \in N(i)} x_j \right) \left( 1 - \prod_{k \in N_2(i)} x_k \right) \),

\[ E(|Y|) = \sum_{i \in V} (1 - x_i) \prod_{j \in N(i)} (1 - x_j), \text{ and} \]

\[ E(|I|) \geq \sum_{i \in V} \frac{1}{1 + d_i} (1 - x_i) \left( \prod_{j \in N(i)} (1 - x_j) \right) \left( \prod_{k \in N_2(i)} (1 - x_k) \right). \]
Proof. $E(|X|) = \sum_{i \in V} P(i \in X) = \sum_{i \in V} x_i$.

$E(|X'|) = \sum_{i \in V} P(i \in X') = \sum_{i \in V} P(i \in X \cap N(i) \subseteq X \cap N(i) \cap (X \setminus X') \neq \emptyset)$

$= \sum_{i \in V} P(i \in X) P(N(i) \subseteq X) P(N(i) \cap (X \setminus X') \neq \emptyset | i \in X \cap N(i) \subseteq X)$

$= \sum_{i \in V} x_i \left( \prod_{j \in N(i)} x_j \left( 1 - P(N(i) \subseteq X' | i \in X \cap N(i) \subseteq X) \right) \right)$

$= \sum_{i \in V} \left( \sum_{i \in V} x_i \left( \prod_{j \in N(i)} x_j \right) \left( 1 - P(N_2(i) \subseteq X) \right) \right)$

$= \sum_{i \in V} \left( 1 - x_i \right) \left( \prod_{j \in N(i)} (1 - x_j) \right)$.

A lower bound on $|I|$ (see [1, 9, 2, 6]) is given by the following inequality $|I| \geq \sum_{i \in V} \frac{1}{1 + d_i}$. For $i \in V(G)$ define the random variable $Z_i$ with $Z_i = \frac{1}{1 + d_i}$ if $i \in Y'$ and $Z_i = 0$ if $i \notin Y'$. Hence,

$$E(|I|) \geq E\left( \sum_{i \in V} Z_i \right) = \sum_{i \in V} E(Z_i) = \sum_{i \in V} \frac{1}{1 + d_i} P(i \in Y')$$

$$\geq \sum_{i \in V} \frac{1}{1 + d_i} P(i \notin X \cap N(i) \cap X = \emptyset \cap N(i) \cap Y \neq \emptyset).$$

Because $d_i \geq 1$, $N_2(i) \cap X = \emptyset$ implies $N(i) \cap Y \neq \emptyset$. Hence,

$$E(|I|) \geq \sum_{i \in V} \frac{1}{1 + d_i} P(i \notin X \cap N(i) \cap X = \emptyset \cap N_2(i) \cap X = \emptyset)$$

$$\geq \sum_{i \in V} \frac{1}{1 + d_i} P(i \notin X) P(N(i) \cap X = \emptyset) P(N_2(i) \cap X = \emptyset)$$

$$\geq \sum_{i \in V} \frac{1}{1 + d_i} \left( 1 - x_i \right) \left( \prod_{j \in N(i)} (1 - x_j) \right) \left( \prod_{k \in N_2(i)} (1 - x_k) \right).$$
From Lemma 4 and Lemma 5 we have $\gamma \leq g_G(x_1, \ldots, x_n) \leq f_G(x_1, \ldots, x_n)$. Let $D^*$ be a minimum dominating set of $G$ and let $y_i = 1$ if $i \in D^*$ and $y_i = 0$ if $i \notin D^*$. Then $y_i \prod_{j \in N(i)} y_j = 0$ and $(1 - y_i) \prod_{j \in N(i)} (1 - y_j) = 0$ for every $i \in V$. $\gamma = |D^*| = \sum_{i \in V} y_i = g_G(y_1, \ldots, y_n) = f_G(y_1, \ldots, y_n)$, and the proof of Theorem 1 is complete.

**Proof of Theorem 2.** Given a graph $H$ on $n_H$ vertices with $m_H$ edges, there is an $O(n_H + m_H)$-algorithm $A$ finding an independent set of $H$ with cardinality at least $\sum_{y \in V(H)} \frac{1}{1 + d_H(y)}$, where $d_H(y)$ is the degree of $y \in V(H)$ (see [2]).

First we present an algorithm that constructs a set $D \subseteq V$.

**Algorithm**

**INPUT:** a graph $G \in \Gamma$ on $V = \{1, \ldots, n\}$, $x_1, \ldots, x_n \in [0, 1]$

**OUTPUT:** $D$

(1) For $l = 1, \ldots, n$ do if $\frac{\partial g_G(x_1, \ldots, x_n)}{\partial x_l} \geq 0$ then $x_l := 0$ else $x_l := 1$.

(2) $X := \{l \in \{1, \ldots, n\} | x_l = 1\}$. Calculate $X''$, $Y$, $Y'$, and $I$ using $A$.

(3) $D := (X \setminus X'') \cup (Y' \setminus I)$.

**END**

Let $g^* = g_G(x_1, \ldots, x_n)$, where $(x_1, \ldots, x_n)$ is the input vector. Note that the function $g_G$ is linear in each variable. Thus, in step (1), for fixed $x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n$ we always choose $x_l$ in such a way that the value of $g_G(x_1, \ldots, x_n)$ is not increased. Hence, $x_l \in \{0, 1\}$ for $l = 1, \ldots, n$ and $g_G(x_1, \ldots, x_n) \leq g^*$ after step (1) of the algorithm. With Lemma 3, $D$ is a dominating set, and with $|S| = E(|S|)$ for a deterministic set $S$ and Lemma 5, $|D| \leq g^*$. It is easy to see that $\frac{\partial g_G(x_1, \ldots, x_n)}{\partial x_l}$ can be calculated in $O(\Delta^4)$ time. Since $G$ has $O(\Delta n)$ edges, the algorithm runs in $O(\Delta^4 n)$ time.

**References**


Received 23 September 2003
Revised 15 June 2004