ON NEUMANN BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS

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Abstract

We provide two existence results for the nonlinear Neumann problem
\[
\begin{aligned}
-\text{div}(a(x)\nabla u(x)) &= f(x,u) \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \), \( a \) is a weight function and \( f \) a nonlinear perturbation. Our approach is variational in character.

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1. Introduction and results

In this paper, we deal with problems of the form
\[
\begin{aligned}
-\text{div}(a(x)\nabla u(x)) &= f(x,u) \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with a \( C^1 \) boundary \( \partial \Omega \), \( f(.,.) \) is a Carathéodory function and \( a(.,) \) is a positive weight on \( \Omega \). Our work is motivated by the results in [1] and [2] concerning the Dirichlet problem. We provide two existence results for \((*)\), the first for a Carathéodory
function \( f \) with sublinear growth at infinity and the other for a continuous function \( f \) which is independent of the space variable. We refer to [6] for a similar result but with a different behavior of \( f \) at infinity and to [4] for an unbounded domain \( \Omega \).

For the first existence result we make the following assumptions:

- \( H(a) \) the weight function \( a : \Omega \rightarrow \mathbb{R} \) is positive a.e. in \( x \in \Omega \) and \( a, a^{-s} \in L^1(\Omega) \) where \( s > \frac{N}{2} \).
- \( H(f) \) \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function (that is, \( f(x, u) \) is measurable in \( x \) for every \( u \in \mathbb{R} \) and continuous in \( u \) for almost every \( x \in \Omega \)) such that
  1. \( |f(x, u)| \leq A, A \in \mathbb{R} \), for every \( u \in \mathbb{R} \) and almost every \( x \in \Omega \).
  2. \( \lim_{u \to \pm \infty} f(x, u) \text{sign} u = f^+(x) \), where \( f^+ \in L^\infty(\Omega) \), \( f^+ \geq 0 \), with a strict inequality holding in a set of positive measure.
  3. \( \limsup_{u \to 0} \frac{F(x, u)}{|u|^2} \leq \theta(x) \) uniformly in \( x \) for almost every \( x \in \Omega \), where \( \theta \in L^\infty(\Omega) \), \( \theta(x) \leq 0 \) with a strict inequality holding in a set of positive measure.

**Remark.** Hypothesis \( H(a) \) implies that the space \( H^1(\Omega, a) = \{ u \in L^2(\Omega) : \int_\Omega a(x)|Du|^2 dx < +\infty \} \) supplied with the norm

\[
\|u\| = \left( \int_\Omega a(x)|Du|^2 dx + \int_\Omega |u|^2 dx \right)^{\frac{1}{2}}
\]

is reflexive. For more details we refer to [5].

Consider the Euler-Lagrange functional associated with (\(
\Phi(u) := \frac{1}{2} \int_\Omega a(x)|\nabla u|^2 dx - \int_\Omega F(x, u) dx,
\)
where

\[
F(x, u) := \int_0^u f(x, t) dt.
\]

It is well known that if the growth of \( f(., .) \) is up to critical, then \( \Phi(., .) \) is a well defined \( C^1 \) functional on \( H^1(\Omega, a) \).

We need two auxiliary lemmas.

**Lemma 1.** \( \Phi(., .) \) satisfies the Palais-Smale condition.
Proof. Suppose not. Then, there exists a sequence \( \{u_n\}_{n \in \mathbb{N}} \) in \( H^1(\Omega, a) \) such that \( |\Phi(u_n)| \leq c, \ c \in \mathbb{R} \), and \( \Phi'(u_n) \to 0 \) and \( \|u_n\| \to +\infty \). Let \( y_n = \frac{u_n}{\|u_n\|} \). By passing to a subsequence if necessary, we may assume that \( y_n \rightharpoonup y \) weakly in \( H^1(\Omega, a) \), \( y_n \to y \) strongly in \( L^2(\Omega) \) and \( y_n(x) \to y(x) \) a.e. Since \( |\Phi(u_n)| \leq c \) we have that

\[
\left| \frac{1}{2} \int_{\Omega} a(x) |Dy_n|^2 \, dx - \frac{1}{\|u_n\|^2} \int_{\Omega} \int_{0}^{u_n} f(x, s) \, ds \, dx \right| \leq \frac{c}{\|u_n\|^2}.
\]

By the Sobolev embedding

\[
\left| \int_{\Omega} \int_{0}^{u_n} f(x, s) \, ds \, dx \right| \leq A \int_{\Omega} |u_n| \, dx \leq c_1 \|u_n\|_2 \leq c_2 \|u_n\|,
\]

\( c_1, c_2 \in \mathbb{R} \). Therefore

\[
\int_{\Omega} a(x) |Dy_n|^2 \to 0.
\]

Exploiting the lower semicontinuity of the norm of \( H^1(\Omega, a) \) we deduce that

\[
\int_{\Omega} a(x) |Dy|^2 \, dx = 0,
\]

so \( y = \xi, \ \xi \neq 0 \). Consequently, \( |u_n(x)| \to +\infty \) a.e. in \( \Omega \). Since \( \Phi'(u_n) \to 0 \), there exists a decreasing sequence \( \{\varepsilon_n\}_{n \in \mathbb{N}} \) of positive real numbers such that \( \varepsilon_n \to 0 \) and

(1) \[
\langle \Phi'(u_n), v \rangle \leq \varepsilon_n \|v\|
\]

for every \( n \in \mathbb{N} \) and every \( v \in H^1(\Omega, a) \). By taking \( v = u_n \) and dividing (1) by \( \|u_n\| \) we get

\[
\left| \int_{\Omega} a(x) |Dy_n|^2 \, dx - \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} u_n \, dx \right| \leq \varepsilon_n.
\]
\[ 0 = \liminf \int_\Omega \frac{f(x, u_n)u_n}{\|u_n\|} \, dx = \liminf \int_\Omega f(x, u_n) \text{sign}(u_n) \frac{|u_n|}{\|u_n\|} \, dx \]

\[ \geq \int_\Omega f^+ |\xi| \, dx, \]

(2)

a contradiction. Therefore the sequence \( \{u_n\}_{n \in \mathbb{N}} \) is bounded. So there exists \( u \in H^1(\Omega, a) \) such that, up to a subsequence, \( u_n \rightharpoonup u \) weakly in \( H^1(\Omega, a) \), \( u_n \rightarrow u \) strongly in \( L^2(\Omega) \) and \( u_n(x) \rightarrow u(x) \) a.e. By taking \( v = u_n - u \) in (1) we get

\[
\left| \int_\Omega a(x) Du_n (Du_n - Du) dx - \int_\Omega f(x, u_n)(u_n - u) dx \right| \leq \varepsilon_n \|u_n - u\|.
\]

Since \( f(\cdot, \cdot) \) is bounded \( \int_\Omega f(x, u_n)(u_n - u) dx \rightarrow 0 \), and consequently \( \int_\Omega a(x) Du_n (Du_n - Du) dx \rightarrow 0 \) as \( n \rightarrow +\infty \). Thus

\[
\int_\Omega a(x) |Du_n - Du|^2 \, dx
\]

\[
= \int_\Omega a(x) Du_n (Du_n - Du) dx - \int_\Omega a(x) Du (Du_n - Du) dx \rightarrow 0,
\]

so \( u_n \rightarrow u \) strongly in \( H^1(\Omega, a) \). \( \blacksquare \)

**Lemma 2.** There exist \( \rho, \eta > 0 \) such that \( \Phi(u) > \eta \) for every \( u \in H^1(\Omega, a) \) with \( \|u\| = \rho \).

**Proof.** We will show that if \( \|u_n\| = \rho_n \downarrow 0 \), then \( \Phi(u) > 0 \). For if this is not true, then there exists a sequence \( \{u_n\}_{n \in \mathbb{N}} \) such that \( \|u_n\| = \rho_n \downarrow 0 \) and \( \Phi(u) \leq 0 \). Thus

\[
\frac{1}{2} \int_\Omega a(x) |Du_n|^2 \, dx - \int_\Omega \int_0^{u_n} f(x, s) ds \, dx \leq 0.
\]

Dividing with \( \|u_n\|^2 \) we get

\[
\frac{1}{2} \int_\Omega a(x) |Dy_n|^2 - \frac{1}{\|u_n\|^2} \int_\Omega \int_0^{u_n} f(x, s) ds \, dx \leq 0,
\]

(3)
where \( y_n = \frac{u_n}{\|u_n\|} \). Note that, because of \( H(f)\text{(iii)} \), for \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( |u| < \delta \), then \( F(x, u) \leq (\theta(x) + \varepsilon)|u|^2 \). Also, \( H(f)\text{(i)} \) implies that \( |F(x, u)| \leq A|u| \) a.e. in \( x \in \Omega \), for every \( u \in \mathbb{R} \). Thus,

\[
|F(x, u)| \leq (\theta(x) + \varepsilon)|u|^2 + \beta|u|^{2^*},
\]

where \( \beta \geq A\delta^{1-2^*} - (||\theta||_\infty + \varepsilon)\delta^{2-2^*} \). From [3] and [4] we deduce that

\[
\frac{1}{2} \int_\Omega a(x) |Dy_n|^2 \leq \int_\Omega (\theta(x) + \varepsilon)|y_n|^2 \, dx + \beta \int_\Omega |u|^2 \, dx
\]

\[
\leq \int_\Omega (\theta(x) + \varepsilon)|y_n|^2 \, dx + \beta \|u_n\|^{2^* - 2},
\]

which, in view of \( H(f)\text{(iii)} \), implies that \( \|Dy_n\|_2 \to 0 \). Since the sequence \( \{y_n\}_{n \in \mathbb{N}} \) is bounded in \( H^1(\Omega, a) \), there exists \( y \in H^1(\Omega, a) \) such that \( y_n \rightharpoonup y \) weakly in \( H^1(\Omega, a) \). Therefore \( \|Dy\|_2 = 0 \), i.e., \( y(x) = \kappa \in \mathbb{R}, \kappa \neq 0 \). But then (5) implies that

\[
\int_\Omega (\theta(x) + \varepsilon) \, dx \geq 0 \text{ for every } \varepsilon > 0,
\]

a contradiction.

We can now state our first existence result.

\textbf{Theorem 1.} Assume that hypotheses \( H(a) \) and \( H(f) \) are satisfied Then problem \((\ast)\) has a solution.

\textbf{Proof.} We intend to use the mountain pass theorem [7]. In view of the above Lemmas, it remains to show that there exists a point \( e \in H^1(\Omega, a) \) such that \( \Phi(e) < 0 \). Note that if we take \( u_\beta(x) := \beta \in \mathbb{R} \) for every \( x \in \Omega \), then

\[
\Phi(u_\beta) = -\int_\Omega \int_0^{u_\beta} f(x, s) \, ds \, dx = -\int_\Omega \int_0^{\beta} f(x, s) \, ds \, dx \to -\infty
\]

as \( \beta \to +\infty \) because of \( H(f)\text{(ii)} \) and the result follows by taking \( \beta \) large enough.

\]
If we assume that \( f \) depends only on \( u \), then we can remove the hypothesis on its growth. The proof of this result is inspired by [2]. So consider the problem

\[
\begin{array}{ll}
-\text{div}(a(x)\nabla u) = f(u) & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega.
\end{array}
\]

We make the following assumptions:

\( Hc(a) \) \( a : \mathbb{R} \to \mathbb{R} \) is a function in \( L^\infty(\Omega) \) such that \( a(x) \geq \sigma > 0 \) a.e. in \( x \in \Omega \).

\( Hc(f) \) \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function such that

(i) \( \lim_{|u| \to +\infty} \frac{f(u)}{u} = 0 \)

(ii) \( B = \limsup_{u \to -\infty} G(u) < 0 \) and \( \Gamma = \liminf_{u \to +\infty} G(u) > 0 \), where \( G(u) = \begin{cases} 
\frac{2}{u} \int_0^u f(s)ds - f(u) & \text{if } u \neq 0 \\
f(0) & \text{if } u = 0.
\end{cases} \)

**Lemma 3.** Assume that hypotheses \( Hc(a) \) and \( Hc(f) \) are satisfied. Then \( \Phi(.) \) satisfies the Palais-Smale condition.

**Proof.** Suppose that \( \{u_n\}_{n \in \mathbb{N}} \) is a sequence in \( H^1(\Omega, a) \) such that \( |\Phi(u_n)| \leq c, c \in \mathbb{R}, \) and \( \Phi(u_n) \to 0 \). As in the proof of Lemma 1 we will show first that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded. So assume that \( \|u_n\| \to +\infty \) and let \( y_n = \frac{u_n}{\|u_n\|} \). By passing to a subsequence if necessary, we may assume that \( y_n \to y \) strongly in \( L^2(\Omega) \) and \( y_n(x) \to y(x) \) a.e. Since \( |\Phi(u_n)| \leq c \) we have that

\[
\left| \frac{1}{2} \int_{\Omega} a(x)|Du_n|^2 \, dx - \int_{\Omega} \int_0^{u_n} f(s)ds \, dx \right| \leq c.
\]

By dividing this inequality with \( \|u_n\|^2 \) we get

\[
\left| \frac{1}{2} \int_{\Omega} a(x)|Du_n|^2 \, dx - \frac{1}{\|u_n\|^2} \int_{\Omega} \int_0^{u_n} f(s)ds \, dx \right| \leq \frac{c}{\|u_n\|^2}.
\]
In view of $Hc(f)(i)$,

$$\int_{\Omega} a(x) |Dy_n|^2 \to 0 \text{ as } n \to +\infty,$$

which implies that $y = \xi \in \mathbb{R}$, $\xi \neq 0$. Consequently, $|u_n(x)| \to +\infty$ as $n \to +\infty$ a.e. in $\Omega$. So for $\varepsilon, \delta > 0$, by Egoroff’s theorem, there exists a measurable subset $\Sigma$ of $\Omega$ and $n_0 \in \mathbb{N}$ such that

$$\mu(\Omega \setminus \Sigma) < \delta \quad \text{and} \quad |y_n(x) - \xi| < \varepsilon \quad \text{for } x \in \Sigma \text{ and } n > n_0. \quad (6)$$

Hence, for any $\zeta \in \mathbb{R}$, we have

$$\mu\{x \in \Omega : |u_n(x)| \leq \zeta\} = \mu\{x \in \Omega \setminus \Sigma : |u_n(x)| \leq \zeta\} + \mu\{x \in \Sigma : |u_n(x)| \leq \zeta\} \leq \delta + \mu\{x \in \Sigma : |u_n(x)| \leq \zeta\},$$

which combined with [6] yields

$$\lim_{n \to +\infty} \mu\{x \in \Omega : |u_n(x)| \leq \zeta\} = 0.$$

Because of our hypotheses on $\{u_n\}_{n \in \mathbb{N}}$ there holds

$$\lim_{n \to +\infty} \frac{\langle \Phi'(u_n), u_n \rangle - 2\Phi(u_n)}{\|u_n\|} = 0.$$

Thus

$$\lim_{n \to +\infty} \frac{2 \int_{\Omega} \int_0^{u_n} f(s)dsdx - \int_{\Omega} f(u_n)u_n dx}{\|u_n\|} = \lim_{n \to +\infty} \int_{\Omega} G(u_n) \frac{u_n}{\|u_n\|} dx = 0.$$
such that
\[ G(u) \geq \Gamma - \varepsilon = \Gamma_\varepsilon \text{ if } u > \zeta, \quad G(u) \leq B + \varepsilon = B_\varepsilon \text{ if } u < -\zeta \]
\[ |G(u)| \leq \eta \text{ if } |u| \leq \zeta, \]
where
\[ B_\varepsilon = \begin{cases} \limsup_{u \to -\infty} G(u) - \varepsilon & \text{if } \limsup_{u \to -\infty} G(u) > -\infty \\ -\frac{1}{\varepsilon} & \text{otherwise}, \end{cases} \]
and
\[ \Gamma_\varepsilon = \begin{cases} \liminf_{u \to +\infty} G(u) + \varepsilon & \text{if } \liminf_{u \to +\infty} G(u) < +\infty \\ \frac{1}{\varepsilon} & \text{otherwise}. \end{cases} \]
Assume first that \( \xi > 0 \). Note that
\[
\int_\Omega G(u_n) \frac{u_n}{\|u_n\|} dx = \int_{|u_n| \leq \zeta} G(u_n) \frac{u_n}{\|u_n\|} dx + \int_{u_n > \zeta} G(u_n) \frac{u_n}{\|u_n\|} dx + \int_{u_n < -\zeta} G(u_n) \frac{u_n}{\|u_n\|} dx.
\]
Since
\[
\left| \int_{|u_n| \leq \zeta} G(u_n) \frac{u_n}{\|u_n\|} dx \right| \leq \frac{\eta \xi \mu \{x \in \Omega : |u_n(x)| \leq \zeta\}}{\|u_n\|} \to 0,
\]
\[
\liminf_{n \to +\infty} \int_{u_n > \zeta} G(u_n) \frac{u_n}{\|u_n\|} dx \geq 0
\]
and
\[
\liminf_{n \to +\infty} \int_{u_n < -\zeta} G(u_n) \frac{u_n}{\|u_n\|} dx \geq \int_{u_n > \zeta} \liminf_{n \to +\infty} G(u_n) \frac{u_n}{\|u_n\|} dx = \liminf_{u \to +\infty} G(u) \mu(\Omega) \xi > 0,
\]
we get
\[
0 = \liminf_{n \to +\infty} \int_{\Omega} \frac{u_n}{\|u_n\|} \, dx \geq \liminf_{n \to +\infty} \int_{u_n > \zeta} G(u_n) \frac{u_n}{\|u_n\|} \, dx > 0,
\]
a contradiction. Similarly for \(\xi < 0\). Thus \(\{u_n\}\) is bounded. We can now proceed as in the previous theorem.

We denote by \(X_1\) the subspace of \(H^1(\Omega, a)\) consisting of the constant functions and by \(X_2 = \{u \in H^1(\Omega, a) : \text{there exists } v \in H^1(\Omega, a) \text{ such that } u(x) = v(x) - \frac{1}{\mu(\Omega)} \int_{\Omega} v \, d\mu\}\) its complement. Then \(H^1(\Omega, a) = X_1 \oplus X_2\).

For \(u \in H^1(\Omega, a)\) let
\[
\bar{u} = \frac{1}{\mu(\Omega)} \int_{\Omega} u \, d\mu.
\]

Lemma 4. (i) \(\Phi(h) \to -\infty\) as \(|h| \to +\infty\) for \(h \in X_1\), and
(ii) \(\Phi\) is bounded from below in \(X_2\).

Proof. (i) Let us assume that there exists a sequence \(\{\gamma_n\}_{n \in \mathbb{N}}\) of real numbers such that \(|\gamma_n| \to +\infty\) and \(|\Phi(\gamma_n)| \leq c\) for some \(c \in \mathbb{R}\). Suppose first that \(\gamma_n \to +\infty\). As in [2] we can show that
\[
\frac{F(u)}{u} \geq \Gamma \epsilon
\]
for \(u > \zeta\). Thus
\[
0 = \limsup_{n \to -\infty} \frac{\Phi(\gamma_n)}{\gamma_n} = -\liminf_{n \to -\infty} \int_{\Omega} \frac{F(\gamma_n)}{\gamma_n} \, dx
\]
\[
\leq -\int_{\Omega} \Gamma \epsilon \, dx,
\]
a contradiction. Similarly for \(\gamma_n \to -\infty\). So (i) holds.

To prove (ii) we proceed as follows
\[
\Phi(u - \bar{u}) - \int_{\Omega} a(x) |\nabla u|^2 \, dx = -\int_{\Omega} \int_{0}^{u - \bar{u}} f(s) \, ds \, dx
\]
\[
= -\int_{\Omega} \int_{0}^{\zeta} f(s) \, ds \, dx - \int_{\Omega} \int_{\zeta}^{u - \bar{u}} f(s) \, ds \, dx
\]
\[
\geq C(\zeta, \eta) - \varepsilon \int_{\Omega} |u - \bar{u}| \, dx \quad \text{(by Hcf (i))}
\]
\[
\geq C(\zeta, \eta) - d_1 \|u - \bar{u}\|_2,
\]
where $C(\zeta, \eta)$ is a constant which depends only on $\zeta$, $\eta$ and $d_1$ is a constant which depends only on $\varepsilon$ and $\Omega$. Thus, by the Poincare-Wirtinger inequality

$$\Phi(u - \overline{u}) \geq \int_{\Omega} a(x) |\nabla u|^2 \, dx + C(\zeta, \eta) - d_2 \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2},$$

where $d_2$ is a constant which depends only on $\varepsilon$, $a$ and $\Omega$, proving (ii).

We can now apply the saddle point theorem, see [7], to show the following

**Theorem 2.** Suppose that hypotheses $Hc(a)$ and $Hc(f)$ hold. Then (**) has a solution.

**References**


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