SOLUTION OF FREDHOLM INTEGRODIFFERENTIAL EQUATION FOR AN INFINITE ELASTIC PLATE

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Abstract
Many authors discussed the problem of an elastic infinite plate with a curvilinear hole, some of them considered this problem in z-plane and the others in the s-plane. They obtained an exact expression for Goursat’s functions for the first and second fundamental problem. In this paper, we use the Cauchy integral method to obtain a solution to the first and second fundamental problem by using a new transformation. Some applications are investigated and also some special cases are discussed.

Keywords: integrodifferential equation, Cauchy method, complex variable, infinite plate, curvilinear hole.

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1. Introduction

Many authors discussed boundary value problems for isotropic homogeneous infinite plates. Some authors [4, 14, 15] used Lauran’s theorem to express each complex potential function as a power series, others [1, 3, 5, 6, 8, 9, 10, 11] used the complex variable method of Cauchy’s integrals.

From [13] we know that the first and second fundamental problem in the plane theory of elasticity are equivalent to finding two analytic functions $\phi(z)$ and $\psi(z)$, these function must satisfy the boundary condition

$$k\phi(t) - \bar{t}\phi'(t) - \psi(t) = f(t).$$
We deal with the first fundamental problem when \( k = -1 \) and \( f(t) \) is a given function of stresses; while in the case of the second fundamental problem when \( k = \frac{\lambda + 3\mu}{\lambda + \mu} > 1 \), \( f(t) = 2\mu g(t) \) is a given function of displacement where \( \lambda \) and \( \mu \) are called Lame’s constants.

Muskhellishvili in [13] studied the problem by using conformal mapping \( z = cw(\xi), \ c > 0 \), and \( w'(z) \) does not vanish or becomes \( \infty \) for \( |\xi| > 1 \), then the infinite region is exterior to the unit circle \( \gamma \). From [4, 13, 15] the two complex functions of potentials are

\[
\phi(z) = \frac{X + iY}{2\pi(1 + \chi)} \ln \xi + c\Gamma \xi + \phi_0(\xi)
\]

\[
\psi(z) = \frac{\chi(X + iY)}{2\pi(1 + \chi)} \ln \xi + c\Gamma \xi + \psi_0(\xi)
\]

and also the stress plane components are given by

\[
\sigma_{xx} + \sigma_{yy} = 4 \text{Re}\{\phi'(z)\},
\]

\[
\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = 2 \left[ z\phi''(z) + \psi'(z) \right],
\]

where \( X, Y \) being the components of the resultant vector of all external forces. \( \Gamma^*, \Gamma \) are constants and \( \chi \) in the Muskhelishvili’s constant.

2. Mapping function

Many different mapping functions were used in [1, 5, 8, 9, 10, 13] to solve the first and second fundamental problem of an infinite plate with a curvilinear hole. In [3], Abdou and El-Bary used a general transformation with \( n \)-poles to solve the same problem. In this paper, we use a more general transformation with many strong poles to solve the same boundary value problem but with different type of singularities as

\[
z = cw(\xi) = \frac{\xi + m\xi^{-1}}{(1 - n_1\xi^{-1})^2(1 - n_2\xi^{-1})}, \quad n_1 \neq n_2.
\]
3. Method of solution

We use the mapping function (2.1), where $m, n_1$ and $n_2$ are real parameters. Now, if $z = cw(\xi)$, therefore $z' = cw'(\xi)$

\[(3.1) \quad \frac{z}{z} = \frac{w(\xi)}{w'(\xi)} = \alpha(\xi) + \beta(\xi),\]

where $\alpha(\xi)$ is a singular term, while $\beta(\xi)$ is a regular function for $|\xi| > 1$.

We can rewrite equation (3.1) as

\[(3.2) \quad \frac{w(\xi)}{w'(\xi)} = \frac{h_1}{(n_1 - \xi)^2} + \frac{h_2}{(n_2 - \xi)} + \beta(\xi),\]

where $h_1$ and $h_2$ are constants and calculated as

\[
h_1 = \frac{1}{a} \sum_{i=1}^{5} L_i n_1^{i-1} \left[ ab \sum_{i=1}^{5} L_i n_1^{i-1} + c + d - Ea \sum_{i=1}^{4} i L_i n_1^{i-1} + \sum_{i=1}^{5} L_i n_1^{i-1} \right],\]

and

\[
h_2 = \frac{(n_2^4 + mn_1^3)(1 - n_1 n_2^2)(1 - n_2^2)^2}{a^2 \sum_{i=1}^{5} L_i n_2^{i-1}},\]

where $L_1 = 1$, $L_2 = -2(2n_1 + n_2)$, $L_3 = -m(2n_1 + n_2) + 3n_1(2n_2 + n_1)$, $L_4 = n_2^2(4m - n_2) + 8n_1 n_2 m$, and $L_5 = -3mn_2 n_1^2$.

and $a = n_1 - n_2$

\[
b = (4n_1^3 - 3mn_1^2)(1 - n_1^5)(1 - n_1 n_2)^2,\]

\[
c = -4n_1^4(n_1^4 + mn_1^3)(1 - n_1 n_2)^2,\]

\[
d = -2n_2(1 - n_1 n_2)(1 - n_1^5)(n_1^4 + mn_1^3),\]

and $E = (n_1^4 + mn_1^3)(1 - n_1^5)(1 - n_1 n_2)^2$. 


By using (3.1) in (1.1) we have

\[(3.6) \quad k\phi(\sigma) - \frac{w(\sigma)}{w'(\sigma)}\phi'(\sigma) - \psi(\sigma) = f(\sigma) .\]

Also, using (3.2) in (3.6) we obtain

\[(3.7) \quad k\phi(\sigma) - \alpha(\sigma)\phi'(\sigma) - \psi_*(\sigma) = f_*(\sigma) ,\]

where,

\[\psi_*(\sigma) = \psi(\xi) + \beta(\xi)\phi'(\xi),\]

\[f_*(\xi) = F(\xi) - c\Gamma\xi + c\Gamma^r\xi^{-1} + N(\xi) \left[ \alpha(\xi) + \beta(\xi) \right],\]

\[N(\xi) = CT - \frac{X - iY}{2\pi(1 - \chi)}\xi, \text{ and}\]

\[F(\xi) = f(t) .\]

Multiplying both sides of (3.7) by \(\frac{d\sigma}{2\pi(\sigma - \xi)}\) and integrating on \(\gamma\) we get

\[(3.8) \quad k\phi(\xi) + \frac{1}{2\pi i} \int_\gamma \frac{\alpha(\sigma)\phi'(\sigma)}{\sigma - \xi} d\sigma = -c\Gamma^r\xi^{-1} - \frac{h_1(X - iY)}{2\pi(1 + \chi)(n_1 - \xi)} + \frac{h_1N(n_1)}{(n_1 - \xi)^2} + \frac{h_2N(n_2)}{n_2 - \xi} - A(\xi) ,\]

where \(A(\xi) = -\frac{1}{2\pi i} \sum_{\eta = 0}^{\infty} \xi^{-\eta - 1} \int_\eta \sigma F(\sigma) d\sigma .\)

Using [4, 5, 8, 13], we have

\[(3.9) \quad \frac{1}{2\pi i} \int_\gamma \frac{\alpha(\sigma)\phi'(\sigma)}{\sigma - \xi} d\sigma = ch_1 \left[ \frac{b_1}{n_1 - \xi} + \frac{b_{11}}{(n_1 - \xi)^2} + \frac{ch_2b_2}{n_2 - \xi} \right],\]

where \(b_1, b_{11},\) and \(b_2\) are complex constants to be determined.
In [2, 7, 11, 12] authors solved some problems similar to (3.8) with different notations and different kernels.

Use (3.9) in (3.8), we get

\[-k \phi(\xi) = A(\xi) + c \Gamma^* \xi^{-1} + \frac{h_1}{n_1 - \xi} \left[ cb_1 + \frac{X - iY}{2\pi(1 + \chi)} \right] + \frac{h_1}{(n_2 - \xi)^2} \left[ cb_{11} + N(n_1) \right] + \frac{h_2}{n_2 - \xi} \left[ cb_2 + N(n_2) \right].
\]

Differentiating (3.10) with respect to \(\xi\), we have three equations in \(b_1, b_{11}\), and \(b_2\) and by solving this system we obtain

\[b_1 = \frac{1}{H^-} \left[ 2\nu_1 ReE_2 - \eta_1^+ ReE_1 \right] + \frac{i}{H^+} \left[ \eta_1^- ImE_1 + 2\nu_1 ImE_2 \right]\]

\[b_{11} = \frac{1}{H^-} \left[ \eta_2^+ ReE_2 - h_1 \nu_1^2 ReE_1 \right] + \frac{i}{H^+} \left[ \eta_2^- ImE_2 - h_1 \nu_1^2 ImE_1 \right]\]

\[b_2 = \frac{kE_3 - \nu_2 h_2 E_2}{c \left( k^2 - h_2^2 \nu_2^2 \right)}, \quad \nu_j = \frac{n_j}{(1 - n_j)^2} \quad (j = 1, 2),
\]

where

\[E_1 = A'(n_1^{-1}) - 2n_1 c \Gamma^* + \frac{h_1 v(X + iY)}{2\pi(1 + \chi)} - 3h_1 v \nu_1 N(n_1)\]

\[E_2 = A'(n_1^{-1}) + cn_1^2 \Gamma^* - \frac{h_1 v(X + iY)}{2\pi(1 + \chi)} - 2h_1 \nu_1^3 N(n_1)\]

\[E_3 = A'(n_1^{-1}) + cn_2^2 \Gamma^* - ch_2 \nu_2 + \frac{h_2 \nu_2 n_2 (X + iY)}{2\pi(1 + \chi)} - ch_1 \nu_1 n_2 (X + iY)\]

\[v = 2\nu_1 \left[ \frac{1}{n_1^2} - \nu_1^2 \right].
\]

Also

\[H^\pm = \frac{c}{k} \left[ k \pm 2h_1 \nu_1 \right] \left[ k \pm \frac{2h_1 \nu_1}{(1 - n_1^2)} \right] + \frac{6ch_1^2 \nu_1^3}{k(1 - n_1^2)^2}.
\]
and

\[ \eta_i^\pm = c \left[ 1 \pm \frac{2h_i}{k} \nu_i^2 \right], \quad (i = 1, 2). \]

Finally, we have

\[ -k \phi(\xi) = A(\xi) + c\Gamma^* \xi^{-1} + \frac{T_1}{n_1 - \xi} + \frac{T_2}{(n_1 - \xi)^2} + \frac{T_3}{n_2 - \xi}, \]

where

\[ T_1 = h_1 \left[ cb_1 + \frac{X - iy}{2\pi(1 + \chi)} \right], \quad T_2 = h_1 [cb_1 + N(n_1)] \]

and

\[ T_3 = h_2 [cb_2 + N(n_2)]. \]

From the boundary condition we calculate \( \psi(\xi) \) as

\[ \psi(\xi) = \frac{ck\Gamma}{\xi} - \frac{w(\xi^{-1})}{w'(\xi)} \phi_s(\xi) - \frac{h_1 \xi}{(1 - n_1 \xi)} \phi_s(n_1^{-1}) \]

\[ + \frac{h_1 \xi^2}{(1 - n_1 \xi)^2} \phi_s(n_1^{-1}) - \frac{h_2 \xi}{(1 - n_2 \xi)} \phi_s(n_2^{-1}) + B(\xi) - B, \]

where

\[ \phi_s(\xi) = \phi'(\xi) + N(\xi), \]

\[ B(\xi) = \frac{1}{2\pi i} \int \frac{F(\sigma)}{\sigma - \xi} d\sigma, \quad \text{and} \]

\[ B = \frac{1}{2\pi i} \int \frac{F(\sigma)}{\sigma} d\sigma. \]

The stress components are calculated by using (3.13) and (3.14) in (1.4) and (1.5).
References


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