ORTHORINGS

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Abstract

Certain ring-like structures, so-called orthorings, are introduced which are in a natural one-to-one correspondence with lattices with 0 every principal ideal of which is an ortholattice. This correspondence generalizes the well-known bijection between Boolean rings and Boolean algebras. It turns out that orthorings have nice congruence and ideal properties.

Keywords: ortholattice, generalized ortholattice, sectionally complemented lattice, orthoring, arithmetical variety, weakly regular variety, congruence kernel, ideal term, basis of ideal terms, subtractive term.

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1. Introduction

The well-known natural bijective correspondence between Boolean algebras and Boolean rings is widely used in applications, see [1] for details. This correspondence was generalized in different ways thus giving rise to natural connections between certain lattice structures on the one hand and certain ring-like structures on the other hand. On the lattice-theoretical side the following structures were considered: orthomodular lattices ([8] and [18]), ortholattices ([2]), bounded lattices with an involutory antiautomorphism ([9], [10], [11], [12], [13] and [14]), pseudocomplemented semilattices ([5]) and MV-algebras ([6]). The corresponding ring-like structures were called orthomodular Boolean quasirings or orhomodular pseudorings, orthopseudorings, orthopseudosemirings, Boolean quasirings and pseudorings, respectively. (No name was assigned to the ring-like structures induced by pseudocomplemented semilattices.) In [3] the ring-like structures introduced in [2] and [9], respectively, are related to each other. However, each one of the derived ring-like structures considered so far in this context was endowed with a constant 1 which plays a role similar to the unit element in rings. On the other hand, starting with so-called generalized Boolean algebras, one can derive Boolean rings which need not have a unit element (see [1]). A similar approach was used in [7], where so-called generalized orthomodular lattices (introduced by M.F. Janowitz in [17], see also [16]) were considered. Our aim is to investigate ring-like structures (so-called orthorings) which correspond to lattices with 0 such that every principal lattice ideal is an ortholattice. It should be pointed out that though these lattices do not form a variety, the term equivalent orthorings form a variety and hence allow the application of universal algebraic methods and results. Moreover, we are going to show that – in spite of their generality – orthorings have nice properties.

We recall that an ortholattice is an algebra \((L; \lor, \land, ', 0, 1)\) of type \((2, 2, 1, 0, 0)\) such that \((L; \lor, \land, 0, 1)\) is a bounded lattice and \((x')' = x, (x \lor y)' = x' \land y', (x \land y)' = x' \lor y', x \lor x' = 1\) and \(x \land x' = 0\) for all \(x, y \in L\).

2. Orthorings

First we introduce the concept of a generalized ortholattice and distinguish generalized ortholattices from other classes of lattices.
Definition 2.1. (cf., e.g., [15]). A lattice \((L; \lor, \land, 0)\) with 0 is called a \textit{sectionally complemented lattice} if for each \(a \in L\), \(([0, a]; \lor, \land)\) is a complemented lattice, i.e., for every \(a, b \in L\) with \(b \leq a\) there exists an element \(c\) of \(L\) with \(c \leq a\), \(b \lor c = a\) and \(b \land c = 0\).

Definition 2.2. \((L; \lor, \land, (a; a \in L), 0)\) where \((L; \lor, \land, 0)\) is a lattice with 0 is called a \textit{generalized ortholattice} if for each \(a \in L\), \(([0, a]; \lor, \land, a, 0, a)\) is an ortholattice.

Of course, if \((L; \lor, \land, (a; a \in L), 0)\) is a generalized ortholattice, then \((L; \lor, \land, 0)\) is a sectionally complemented lattice.

Example 2.1. The five-element modular non-distributive lattice is sectionally complemented but it cannot be considered as a generalized ortholattice since it has an odd number of elements.

Example 2.2. If \((L; \lor, \land, (a; a \in L), 0, 1)\) is an orthomodular lattice, i.e. an ortholattice satisfying \(y = x \lor (y \land x')\) for all \(x, y \in L\) with \(x \leq y\), then \((L; \lor, \land, (x \mapsto x' \land a; a \in L), 0)\) is a generalized ortholattice.

Example 2.3. The following Hasse diagram shows a non-orthomodular generalized ortholattice:

```
\begin{center}
\begin{tikzpicture}
  \node at (0,0) (1) {$1$};
  \node at (-2,-2) (a) {$a$};
  \node at (-1,-2) (b) {$b$};
  \node at (0,-2) (c) {$c$};
  \node at (1,-2) (d) {$d$};
  \node at (2,-2) (e) {$e$};
  \node at (-2,-4) (a') {$a'$};
  \node at (-1,-4) (b') {$b'$};
  \node at (0,-4) (c') {$c'$};
  \node at (1,-4) (d') {$d'$};
  \node at (2,-4) (e') {$e'$};
  \node at (-4,0) (0) {$0$};

  \draw (1) -- (a); \draw (1) -- (b); \draw (1) -- (c); \draw (1) -- (d); \draw (1) -- (e);
  \draw (a) -- (a'); \draw (b) -- (b'); \draw (c) -- (c'); \draw (d) -- (d'); \draw (e) -- (e');
  \draw (a) -- (c); \draw (b) -- (d); \draw (c) -- (e);
  \draw (a') -- (a''); \draw (b') -- (b''); \draw (c') -- (c''); \draw (d') -- (d''); \draw (e') -- (e'');
  \draw (a'') -- (b''); \draw (c'') -- (d''); \draw (e'') -- (a'');

\end{tikzpicture}
\end{center}
```

Next we introduce ring-like structures corresponding to generalized ortholattices.

Definition 2.3. An \textit{orthoring} is an algebra \((R; +, \cdot, 0)\) of type \((2, 2, 0)\) satisfying
(O1) \( x + y = y + x, \)
(O2) \( x + 0 = x, \)
(O3) \( xy = yx, \)
(O4) \( (xy)z = x(yz), \)
(O5) \( xx = x, \)
(O6) \( x0 = 0, \)
(O7) \( (xy + x) + x = xy, \)
(O8) \( ((x + y) + xy) + xy = x + y, \)
(O9) \( (xy + x)x = xy + x, \)
(O10) \( (x + y)xy = 0, \)
(O11) \( ((x + y) + xy)x = x, \)
(O12) \( ((xy + xz) + xyz)x = (xy + xz) + xyz \)
and
(O13) \( (xyz + x)(xy + x) = xy + x. \)

Remark 2.1. Orthorings \( \mathcal{R} = (R; +, \cdot, 0) \) are of characteristic 2, i.e. \( x + x = 0 \) for all \( x \in R. \)

Proof.

\[
x + x \overset{(O1),(O2)}{=} (0 + x) + x \overset{(O4),(O10)}{=} ((x + x)xx + x) + x \overset{(O3)-(O5)}{=} \\
\overset{(O3)-(O5)}{=} (x(x + x) + x) + x \overset{(O7)}{=} x(x + x) \overset{(O3)-(O5)}{=} (x + x)xx \overset{(O10)}{=} 0.
\]

Now we can state our main result describing a natural bijective correspondence between generalized ortholattices and orthorings.

Theorem 2.1. For fixed set \( L \) the formulas

\[
x + y := (x \land y)^{x\lor y}, \\
xy := x \land y
\]
and

\[ x \lor y := (x + y) + xy, \]
\[ x \land y := xy, \]
\[ xy := x + y \]

induce mutually inverse bijections between the set of all generalized ortholattices on \( L \) and the set of all orthorings on \( L \).

**Proof.** Let \( \mathcal{L} = (L; \lor, \land, (a; a \in L), 0) \) be a generalized ortholattice and put \( x + y := (x \land y)^{x\lor y} \) and \( xy := x \land y \) for all \( x, y \in L \). Let \( x, y, z \in L \). Then

\[ (x + y) + xy = ((x \land y)^{x\lor y} \land x \land y)^{(x \land y)^{x\lor y} \lor (x \land y)} = 0^{x\lor y} = x \lor y, \]
\[ x + 0 = 0^x = x, \]
\[ (xy + x) + x = ((x \land y)^x)^x = x \land y = xy, \]
\[ ((x + y) + xy) + xy = (x \land y)^{x\lor y} = x + y, \]
\[ (xy + x)x = (x \land y)^x \land x = (x \land y)^x = xy + x, \]
\[ (x + y)xy = (x \land y)^{x\lor y} \land x \land y = 0, \]
\[ ((x + y) + xy)x = (x \lor y) \land x = x, \]
\[ ((xy + xz) + xyz)x = ((xy + xz) + (xy)(xz))x = ((x \land y) \lor (x \land z)) \land x =
\]
\[ = (x \land y) \lor (x \land z) = (xy + xz) + (xy)(xz) = (xy + xz) + xyz \text{ and} \]
\[ (xyz + x)(xy + x) = (x \land y \land z)^x \land (x \land y)^x = (x \land y)^x = xy + x. \]

Hence, \( (L; +, \cdot, 0) \) is an orthoring. Moreover,

\[ (x + y) + xy = x \lor y, \]
\[ xy = x \land y \]

and

\[ x \leq y \implies x + y = xy. \]
Therefore, the algebra induced by \((L; +, \cdot, 0)\) according to the formulas given in the theorem coincides with \(L\).

Conversely, let \(\mathcal{R} = (L; +, \cdot, 0)\) be an orthoring and put \(x \vee y := (x + y) + xy\), \(x \wedge y := xy\) and \(x^y := x + y\) for all \(x, y \in L\). Let \((L; \leq)\) denote the poset corresponding to the meet-semilattice \((L; \cdot)\) and \(x, y, z \in L\). Then

\[
\begin{align*}
x(x \vee y) &= x((x + y) + xy) \overset{(O3)}{=} (x + y) + xy \overset{(O11)}{=} x, \text{ i.e. } x \leq x \vee y, \\
y(x \vee y) &= y((x + y) + xy) \overset{(O4)}{=} ((y + x) + yx)y \overset{(O11)}{=} y, \text{ i.e. } y \leq x \vee y, \\
x, y \leq z \text{ implies } (x \vee y)z &= ((x + y) + xy)z \overset{(O3)}{=} ((zx + zy) + zxy)z \\
&\quad \overset{(O4)}{=} (zx + zy) + zxy \overset{(O1)}{=} (x + y) + xy = x \vee y, \text{ i.e. } x \vee y \leq z, \\
x \leq y \text{ implies } x^y y &= (x + y)y \overset{(O9)}{=} (yx + y)y = yx + y \overset{(O3)}{=} x + y = x^y, \\
&\quad \text{ i.e. } x^y \leq y, \\
x \leq y \text{ implies } (x^y)^y &= (x + y) + y \overset{(O3)}{=} (yx + y) + y \overset{(O7)}{=} yx + y \overset{(O3)}{=} x, \\
x \leq y \leq z \text{ implies } y^2 x^z &= (y + z)(x + z) \overset{(O4)}{=} (zyx + z)(zy + z) \\
&\quad \overset{(O4)}{=} zy + z \overset{(O3)}{=} y + z = y^z, \text{ i.e. } y^z \leq x^z \text{ and } \\
x \leq y \text{ implies } x \wedge x^y &= x(x + y) \overset{(O4)}{(O3)}(x + y)xy \overset{(O10)}{=} 0.
\end{align*}
\]

Hence \((L; \vee, \wedge, (\cdot^a; a \in L), 0)\) is a generalized ortholattice. Moreover,

\[
(x \wedge y)x^y = xy + ((x + y) + xy) \overset{(O1)}{=} ((x + y) + xy) + xy \overset{(O8)}{=} x + y \text{ and } \\
x \wedge y = xy.
\]

This shows that the algebra induced by \((L; \vee, \wedge, (\cdot^a; a \in L), 0)\) according to the formulas of the theorem coincides with \(\mathcal{R}\). \(\blacksquare\)
Remark 2.2. If \((L; \lor, \land', 0, 1)\) is a Boolean algebra, then \(x + y = (x \land y)^{x \lor y}\) is the well-known symmetric difference since \((x \land y') \lor (x' \land y) = (x \land y)' \land (x \lor y) = (x \land y)^{x \lor y}\) for all \(x, y \in L\).

Comparing the definition of an orthoring to the definition of a Boolean pseudoring introduced in [7] and comparing the definition of a generalized ortholattice with that of a generalized orthomodular lattice (cf. [17]), we obtain

**Theorem 2.2.** An orthoring \((R; +, \cdot, 0)\) is a Boolean pseudoring if and only if \((x + y)x = x + xy\) and \((xyz + x)y = xyz + xy\) for all \(x, y, z \in R\). A generalized ortholattice \((L; \lor, \land, (^a; a \in L), 0)\) is a generalized orthomodular lattice if and only if \(x^z \land y = x^y\) for all \(x, y, z \in L\) with \(x \leq y \leq z\).

**Proof.** The second assertion can be proved as follows: Let \(\mathcal{L} = (L; \lor, \land, (^a; a \in L), 0)\) be a generalized ortholattice. If \(\mathcal{L}\) is a generalized orthomodular lattice, then \(x^z \land y = x^y\) for all \(x, y, z \in L\) with \(x \leq y \leq z\) according to the definition of a generalized orthomodular lattice. Conversely, if \(x^z \land y = x^y\) for all \(x, y, z \in L\) with \(x \leq y \leq z\), then \(x \lor (y \land x^z) = x \lor x^y = y\) for all \(x, y, z \in L\) with \(x \leq y \leq z\) and, hence, \(([0, z]; \lor, \land, z, 0, z)\) is orthomodular for all \(z \in L\). Therefore, \(\mathcal{L}\) is a generalized orthomodular lattice.

3. Congruence and ideal properties

For an overview on congruence conditions, their characterizations and the theory of ideals in universal algebras, see [4].

A variety is called **arithmetical** if it is both congruence permutable and congruence distributive. A variety with a constant term 0 is called **weakly regular** if any congruence of an algebra belonging to this variety is determined by its 0-class.

It is easy to see that the congruence lattice of an orthoring is a sublattice of the congruence lattice of the corresponding sectionally complemented lattice. Since it is well known that sectionally complemented lattices are arithmetical and weakly regular (see, e.g., [15]), this carries over to orthorings.

Here we will provide a different (and direct) proof of this result.
Theorem 3.1. Orthorings are arithmetical and weakly regular.

Proof. Consider the terms
\[
\begin{align*}
t_1(x, y) & := xy + x, \\
t_2(x, y) & := xy + y, \\
t(x, y, z, u) & := (y + u) + z \quad \text{and} \\
m(x, y, z) & := (xy + yz) + zx.
\end{align*}
\]

We show that \( t_1, t_2 \) and \( t \) satisfy the identities
\[
\begin{align*}
t_1(x, x) & = t_2(x, x) = 0, \\
t(x, y, t_1(x, y), t_2(x, y)) & = x \quad \text{and} \\
t(x, y, 0, 0) & = y
\end{align*}
\]
from which it follows that orthorings are permutative and weakly regular according to Theorem 6.4.11 of [4]. Moreover, we prove that \( m \) is a majority term, i.e. it satisfies
\[
m(x, y, z) = m(x, y, x) = m(y, x, x) = x
\]
from which we obtain that ortholattices are congruence distributive according to Corollary 3.2.4 of [4].

The following calculations yield the desired identities:
\[
\begin{align*}
(x + xy) + xy & = ((x \land y)^x \land (x \land y))^{x \lor (x \land y)} = 0^x = x, \\
t_1(x, x) & = xx + x \quad \text{(O5)} \\
t_2(x, x) & = xx + x \quad \text{(O5)}
\end{align*}
\]

\[
\begin{align*}
t(x, y, t_1(x, y), t_2(x, y)) & = (y + (xy + y)) + (xy + x) \quad \text{(O1),(O3)} \\
& \equiv (O1),(O3) ((yx + y) + (xy + x)) \quad \text{(O7)} \\
& \equiv (O1) yx + (xy + x) \quad \text{(O1),(O3)} \\
& \equiv (O1),(O3) (x + xy) + xy = x,
\end{align*}
\]
Let $V$ be a variety with a constant term 0, $A$ an algebra belonging to $V$ and $B$ a subset of the carrier set of $A$. A term $t(x_1, \ldots, x_n)$ is called an ideal term with respect to the variables $x_i, i \in I$, $(I \subseteq \{1, \ldots, n\})$ if $t(x_1, \ldots, x_n) = 0$ whenever $x_i = 0$ for all $i \in I$. Let $t(x_1, \ldots, x_n)$ be an ideal term with respect to the variables $x_i, i \in I$. $B$ is called closed with respect to $t$ if $t(a_1, \ldots, a_n) \in B$ provided $a_1, \ldots, a_n \in A$ and $a_i \in B$ for all $i \in I$. $B$ is called an ideal of $A$ if $B$ is closed with respect to all ideal terms. A set $T$ of ideal terms is called a basis of ideal terms if a subset of the carrier set of an algebra $C$ belonging to $V$ is an ideal of $C$ whenever it is closed with respect to all ideal terms belonging to $T$. If $\Theta$ is a congruence on $A$, then the congruence class $[0]_\Theta$ of 0 with respect to $\Theta$ is called the congruence kernel of $\Theta$. It is easy to see that every congruence kernel is an ideal. If $V$ has a so-called subtractive term, i.e. a binary term $s$ satisfying $s(x, 0) = x$ and $s(x, x) = 0$, then, conversely, every ideal is a congruence kernel (cf. Theorems 6.6.11 and 10.1.10 of [4]). This is the case with orthorings, because the term $s(x, y) := x + y$ serves as a subtractive term.

Ideals in orthorings can now be characterized as follows:

**Theorem 3.2.** A subset $I$ of the base set $R$ of an orthoring $R$ containing 0 is an ideal of $R$ if and only if $x, y, z, u \in R$ and $xy + x, xy + y, zu + z, zu + u \in I$ together imply $(x + z)(y + u) + (x + z), xyzu + xz \in I$.

**Proof.** This follows from Theorem 10.3.1 of [4] by using the terms introduced in the proof of Theorem 3.1.

**Corollary 3.1.** Every ideal $I$ of an orthoring $R = (R; +, \cdot, 0)$ is the kernel of the congruence $\Theta_I := \{(x, y) \in R \times R \mid xy + x \in I \text{ and } xy + y \in I\}$ on $R$. 

\[
\begin{align*}
t(x, y, 0, 0) &= (y + 0) + 0 \overset{(O2)}{=} y, \\
m(x, x, y) &= (xx + xy) + yx \overset{(O3),(O5)}{=} (x + xy) + xy = x, \\
m(x, y, x) &= (xy + yx) + xx \overset{(O3),(O5)}{=} (xy + xy) + x \overset{(O1),(O2)}{=} x \quad \text{and} \\
m(y, x, x) &= (yx + xx) + xy \overset{(O1),(O3),(O5)}{=} (x + xy) + xy = x.
\end{align*}
\]
Proof. It is almost evident that $I$ is the kernel of $\Theta_I$ where $\Theta_I$ is reflexive and compatible. However, the variety of orthorings is congruence permutable according to Theorem 3.1, and by [19], every compatible reflexive relation on $R$ is a congruence on $R$ (see also Corollary 3.1.13 in [4]).

Finally, we present a finite basis of ideal terms for orthorings:

**Theorem 3.3.** The following terms form a basis of ideal terms for orthorings:

0,

$$(((x+y_1)+y_2)+((z+y_3)+y_4))(x+z)+(((x+y_1)+y_2)+((z+y_3)+y_4)),

(((x+y_1)+y_2)+((z+y_3)+y_4))(x+z)+(x+z),$$

$$(x+y_1+y_2)(z+y_3+y_4)xz+((x+y_1)+y_2)((z+y_3)+y_4),

((x+y_1)+y_2)((z+y_3)+y_4)xz+xz,$$

and

$y_1+y_2$.

**Proof.** This follows from Theorem 10.3.4 of [4] by using the terms introduced in the proof of Theorem 3.1.

References


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