A NOTE ON MINIMALLY 3-CONNECTED GRAPHS *

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Abstract

If $G$ is a minimally 3-connected graph and $C$ is a double cover of the set of edges of $G$ by irreducible walks, then $|E(G)| \geq 2|C| - 2$.

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1. Introduction

A walk $\alpha$ in a simple graph $G$ is a sequence $w_0, w_1, \ldots, w_s$ of vertices of $G$, not necessarily different, such that $w_{i-1}w_i$ is an edge of $G$ for $i = 1, 2, \ldots, s$. An edge $e$ of $G$ is said to be traversed in a walk $\alpha$ if its vertices are consecutive in $\alpha$; an edge may be traversed more than once in a given walk.

A walk $\alpha$ in a graph $G$ is irreducible if $a \neq b$ for every pair $a, b$ of edges which are traversed consecutively in $\alpha$. A set $C$ of irreducible closed walks in a graph $G$ is a walk double cover of $G$ if each edge of $G$ is traversed exactly two times, either once in two different walks in $C$ or twice in the same walk in $C$.

For any simple graph $G$ and any edge $e = uv$ of $G$ we denote by $G - e$ the graph obtained from $G$ by deleting the edge $e$, and by $G \cdot e$ the simple graph obtained from $G$ by identifying the vertices $u$ and $v$ and deleting loops and multiple edges. A minimally 3-connected graph is a 3-connected graph $G$ such that, for every edge $e$ of $G$, the graph $G - e$ is no longer 3-connected.

Whenever possible we follow the terms and notation given in [1]. A wheel $W_t$ is a graph with $t + 1$ vertices, obtained from a cycle $C_t$ with $t$ vertices by adding a new vertex $w$ adjacent to each vertex in $C_t$. The cycle $C_t$ and the vertex $w$ are called the rim and the hub of $W_t$, respectively. In this note we prove the following result.

Theorem 1.1. Let $G$ be a minimally 3-connected graph with $m$ edges. If $C$ is a walk double cover of $G$ with $k$ walks, then $m \geq 2k - 2$. Moreover if $m \leq 2k - 1$, then $G$ is a planar graph and $C$ is the set of planar faces of $G$; in particular if $m = 2k - 2$, then $G$ is a wheel.

2. Proof of Theorem 1.1

The following result due to R. Halin [2] will be used in the proof of Theorem 1.1.

Theorem 2.1. If $e = uv$ is an edge of a minimally 3-connected graph $G$ with $\min\{d(u), d(v)\} \geq 4$, then $e$ lies in no cycle of $G$ of length 3 and $G \cdot e$ is also minimally 3-connected.

For any graph $G$ and any walk double cover $C$ of $G$, we denote by $m(G)$ and by $k(C)$ the number of edges of $G$ and the number of walks in $C$, respectively.
**Remark 1.** Let $G$ be a 3-connected graph and $C$ be a walk double cover of $G$. If two edges $uw$ and $wv$ are consecutive edges in two walks in $C$, then the degree of $w$ is at least 4.

**Proof of Theorem 1.1.** The smallest 3-connected graph is the wheel $W_3$ which is planar and has 6 edges. Since each irreducible walk has at least 3 edges, no walk double cover of $W_3$ has more than 4 walks. Moreover, the only walk double cover of $W_3$ with 4 walks consists of the planar faces of $W_3$.

We proceed by induction assuming $m \geq 7$ and that the result holds for every minimally 3-connected graph with less than $m$ edges.

If $G$ has an edge $e = uv$ with $\min\{d(u), d(v)\} \geq 4$, then by Halin’s theorem, $G \cdot e$ is also minimally 3-connected. Let $C \cdot e$ denote the set of $k$ walks of $G \cdot e$ obtained from the walks in $C$ by contracting the edge $e$.

Also by Halin’s theorem, the edge $e$ lies in no cycle of $G$ of length 3; this implies that all walks in $C \cdot e$ are irreducible. Because $C$ is a walk double cover of $G$ and $e$ is not an edge of $G \cdot e$, $C \cdot e$ is a walk double cover of $G \cdot e$. By induction, $m(G \cdot e) \geq 2k(C \cdot e) - 2$; therefore $m \geq 2k - 1$, since $m(G \cdot e) = m - 1$ and $k(C \cdot e) = k$.

If $m = 2k - 1$, then $m(G \cdot e) = 2k(C \cdot e) - 2$; by induction $G \cdot e$ is a wheel $W_t$ and $C \cdot e$ is the set of planar faces of $W_t$. Let $x$ be the vertex of $W_t$ obtained by identifying $u$ and $v$. Since $u$ and $v$ have degree at least 4 in $G$, the vertex $x$ must be the hub of $W_t$; let $w_0, w_1, \ldots, w_{t-1}$ be the rim of $W_t$.

Since $e$ is in no cycle of $G$ of length 3, $G$ is a graph consisting of the cycle $w_0, w_1, \ldots, w_{t-1}$, the two adjacent vertices $u$ and $v$, and one edge joining each vertex $w_i$ to either $u$ or $v$.

Suppose there are distinct integers $a$, $b$ and $c$ such that $w_a$, $w_{a+1}$ and $w_c$ are adjacent to $u$ in $G$ and $w_{a+1}$, $w_b$ and $w_{c+1}$ are adjacent to $v$ in $G$. The walks $w_a, x, w_{a+1}, w_b, x, w_{b+1}$ and $w_c, x, w_{c+1}$ lie in $C$, since they are faces of $G \cdot e$. This implies that $w_a, u, w_{a+1}, w_b, v, u, w_{b+1}$ and $w_c, u, v, w_{c+1}$ are walks in $C$ which is not possible, since the edge $e = uv$ cannot lie in three walks in $C$.

Therefore there are integers $i$ and $j$ such that $w_i, w_{i+1}, \ldots, w_{j-1}$ are adjacent to $u$ in $G$ and $w_j, w_{j+1}, \ldots, w_{i-1}$ are adjacent to $v$ in $G$. This shows that $G$ is a planar graph.

Since $C \cdot e$ is the set of faces of $G \cdot e = W_t$ and each walk in $C \cdot e$ is either a walk in $C$ or is obtained from a walk in $C$ by contracting the edge $e$, the set $C$ must be the set of faces of $G$. 

We can now assume that each edge of $G$ has at least one end with degree 3. If $C$ contains no cycle of length 3, then $2m \geq 4k$ and $m \geq 2k$. Therefore we can also assume that $C$ contains at least one cycle of length 3. Let $C_3$ be the set of cycles in $C$ of length 3; two cases are considered.

**Case 1.** There is a cycle $\alpha$ in $C_3$ such that no pair of edges of $\alpha$ are traversed consecutively in any other walk in $C$.

Let $u$, $v$ and $w$ be the vertices of $\alpha$. Since each edge of $G$ has an end with degree 3, without loss of generality, we can assume $d_G(u) = d_G(v) = 3$. Let $u_1$ and $v_1$ denote the third vertex of $G$ adjacent to $u$ and the third vertex of $G$ adjacent to $v$, respectively; notice that $u_1 \neq v_1$, since $G$ is 3-connected and has at least 5 vertices.

**Subcase 1.1.** If $d_G(w) = 3$, let $w_1$ denote the third vertex of $G$ adjacent to $w$; as above $u_1 \neq w_1 \neq v_1$. Let $G'$ be the graph obtained from $G$ by contracting the cycle $\alpha$ to a single point $x$. We claim that $G'$ can also be obtained from $G$ by a *delta to wye* transformation (see Figure 1), and therefore it is also a 3-connected graph.

![Figure 1](image.png)

Since $d_G'(x) = 3$ and $d_G'(z) = d_G(z)$ for each vertex $z \neq x$ of $G'$, every edge of $G'$ has an end with degree 3; therefore $G'$ is minimally 3-connected.

Let $C'$ be the set of $k - 1$ walks of $G'$ obtained from the walks in $C \setminus \{\alpha\}$ by contracting the edges $uv$, $vw$ and $wu$. Since no pair of edges of $\alpha$ are consecutive edges in any walk in $C \setminus \{\alpha\}$, all walks in $C'$ are irreducible. Moreover, $C'$ is a walk double cover of $G'$, since $C$ is a walk double cover of $G$ and $uv$, $vw$ and $wu$ are not edges of $G'$.

By induction $m(G') \geq 2k(C') - 2$; hence $m \geq 2k - 1$, since $m(G') = m - 3$ and $k(C') = k - 1$. If $m = 2k - 1$, then $m(G') = 2k(C') - 2$. Again by
induction $G \cdot e$ is a wheel $W_t$ and $C'$ is the set of planar faces of $W_t$. Since $x$ has degree 3 in $G'$, we can assume without loss of generality that $x$ lies in the rim of $G' = W_t$ and that $w_1$ is the hub; this implies that $G$ is a graph as in Figure 2 and therefore it is a planar graph in which $\alpha$ is a face.

Since $C'$ is the set of faces of $G'$ and every walk in $C'$ is either a walk in $C \setminus \{\alpha\}$ or is obtained from a walk in $C \setminus \{\alpha\}$ by contracting some of the edges $uv, vw$ and $wu$, the set $C$ must be the set of planar faces of $G$.

Subcase 1.2. If $d_G(w) \geq 4$, we consider the graph $G \cdot uv$. We claim that $u$ and $v$ cannot be contained in a 3-vertex cut of $G$ and, therefore, $G \cdot uv$ is 3-connected.

Since $d_{G \cdot uv}(x) = 3$ and $d_{G \cdot uv}(z) \leq d_G(z)$ for each vertex $z \neq x$ of $G \cdot uv$, every edge of $G \cdot uv$ has an end with degree 3; therefore $G \cdot uv$ is minimally 3-connected.

Let $C \cdot uv$ be the set of $k-1$ walks of $G \cdot uv$ obtained from the walks in $C \setminus \{\alpha\}$ by contracting the edge $uv$ to a vertex $x$ and substituting each of the edges $uw$ and $vw$ by the edge $xw$. Each walk in $C \cdot uv$ is irreducible, because no pair of edges of $\alpha$ are traversed consecutively in any other walk in $C$. Since $C$ is a walk double cover of $G$ and $uv$ is not an edge of $G \cdot uv$, the set $C \cdot uv$ is a walk double cover of $G \cdot uv$.

By induction $m(G \cdot w) \geq 2k(C \cdot w) - 2$; hence $m \geq 2k - 2$, since $m(G \cdot w) = m - 2$ and $k(C \cdot uv) = k - 1$. If $m \leq 2k - 1$, then $m(G \cdot uv) \leq 2k(C \cdot uv) - 1$; again by induction, $G \cdot uv$ is a planar graph and $C \cdot uv$ is the set of planar faces of $G \cdot uv$.

Since $G \cdot uv$ is 3-connected, there is a planar drawing $\overline{G \cdot uv}$ of $G \cdot uv$ in which $x$ is an interior vertex. Let $R$ be the region formed by the three faces of $\overline{G \cdot uv}$ in which $x$ is a vertex. Since $w, u_1$ and $v_1$ lie in the boundary of $R$
and $x$ is in the interior of $R$, a planar drawing $\overline{G}$ of $G$ can be obtained from $G \cdot uv$ by replacing (within the interior of $R$) the vertex $x$ with two adjacent vertices $u$ and $v$, and the edges $wx, u_1x$ and $v_1x$ with the edges $wu, vw, u_1u$ and $v_1v$ as in Figure 3.

![Figure 3](image1)

Therefore $G$ is a planar graph and $\alpha$ is a face of $G$. Furthermore, $C$ is the set of faces of $G$, since $C \cdot uv$ is the set of planar faces of $G \cdot uv$ and each walk in $C \cdot uv$ is either a walk in $C \setminus \{\alpha\}$ or is obtained from a walk in $C \setminus \{\alpha\}$ by contracting the edge $uv$ to the vertex $x$ and substituting each of the edges $uw$ and $vw$ by the edge $xw$.

If $m = 2k - 2$, then $m(G \cdot uv) = 2k(C \cdot uv) - 2$; again by induction, $G \cdot uv$ is a wheel $W_t$. Since $d_{G \cdot uv}(x) = 3$, we can assume that $x$ lies in the rim of $G \cdot uv$.

If $w$ is the hub of $G \cdot uv$, then $G$ is the wheel $W_{t+1}$, also with hub $w$. If $u_1$ is the hub of $G \cdot uv$, then $G$ is a graph as in Figure 4. Notice that if $t > 3$, then $G - u_1w$ is 3-connected which is not possible since $G$ is minimally 3-connected. Therefore $t = 3$ and $G$ is the wheel $W_4$ with hub $w$. Analogously, if $v_1$ is the hub of $G \cdot uv$, then $G$ is the wheel $W_4$.

![Figure 4](image2)
Case 2. For every cycle $\alpha \in C_3$ there is walk $\sigma_\alpha \neq \alpha$ in $C$ such that two edges of $\alpha$ are traversed consecutively in $\sigma_\alpha$.

For this case, we shall prove that the average length of the walks in $C$ is at least 4 and therefore $2m \geq 4k$ and $m \geq 2k$.

For each $\alpha \in C_3$ let $u_\alpha$, $w_\alpha$ and $v_\alpha$ denote the vertices of $\alpha$. Without loss of generality we assume that $u_\alpha w_\alpha$ and $w_\alpha v_\alpha$ are traversed consecutively in $\sigma_\alpha$. Notice that the walk $\sigma_\alpha$ is uniquely determined since $C$ is a walk double cover of $G$.

By Remark 1, $d_G(w_\alpha) \geq 4$; therefore $d_G(u_\alpha) = d_G(v_\alpha) = 3$, since every edge of $G$ has an end with degree 3. Let $u'_\alpha$ and $v'_\alpha$ denote the third vertex of $G$ adjacent to $u_\alpha$ and the third vertex of $G$ adjacent to $v_\alpha$, respectively.

Again by Remark 1, the edges $w_\alpha u_\alpha$ and $u_\alpha v_\alpha$ are not traversed consecutively in $\sigma_\alpha$; therefore $\sigma_\alpha$ must traverse the edge $u_\alpha u'_\alpha$; analogously $\sigma_\alpha$ traverses the edge $v_\alpha v'_\alpha$. If $u'_\alpha = v'_\alpha$, then $u_\alpha$ and $v_\alpha$ are adjacent only to $u'_\alpha$, to $w_\alpha$ and to each other which is not possible since $G$ is a 3-connected graph with at least 5 vertices; therefore $\sigma_\alpha$ has length at least 5 for each $\alpha \in C_3$. For each $\tau \in C$ let $l(\tau)$ denote the length of $\tau$.

Consider the equivalence relation in $C_3$ given by $\beta \sim \gamma$ if and only if $\sigma_\beta = \sigma_\gamma$. For $\alpha \in C_3$ let $[\alpha]$ denote the equivalence class of $\alpha$.

Let $\beta$ and $\gamma$ be two distinct cycles in $[\alpha]$ and assume, without loss of generality, that the edges $u_\beta w_\beta$, $w_\beta v_\beta$, $u_\gamma w_\gamma$ and $w_\gamma v_\gamma$ are traversed in $\sigma_\alpha = \sigma_\beta = \sigma_\gamma$ in that relative order. The edges $u_\beta w_\beta$ and $w_\beta v_\beta$ are not edges of $\gamma$ since they are traversed in $\beta$ and in $\sigma_\beta \neq \beta$; analogously $u_\gamma w_\gamma$ and $w_\gamma v_\gamma$ are not edges of $\beta$.

Suppose that $w_\beta v_\beta$ and $u_\gamma w_\gamma$ are traversed consecutively in $\sigma_\alpha$. Then $v_\beta = u_\gamma$ and $w_\beta \neq w_\gamma$, since $\sigma_\alpha$ is an irreducible walk. Moreover, $u_\beta = v_\gamma$ since $d_G(v_\beta = w_\gamma) = 3$ and $w_\beta$, $w_\gamma$, $u_\beta$ and $v_\gamma$ are all adjacent to $v_\beta = u_\gamma$. This implies that the vertices $v_\beta = u_\gamma$ and $u_\beta = v_\gamma$ are adjacent in $G$ only to $w_\beta$, to $w_\gamma$, and to each other which is not possible since $G$ is 3-connected and has at least 5 vertices.

Therefore, no edges of two distinct cycles in $[\alpha]$ are traversed consecutively in $\sigma_\alpha$. This implies that $\sigma_\alpha$ has at least $3|\alpha|$ edges.

By the above arguments

$$\frac{l(\sigma_\alpha) + l(\alpha)}{2} \geq \frac{5 + 3}{2} = 4$$
for each \( \alpha \in C_3 \) with \( ||\alpha|| = 1 \), and

\[
\frac{l(\sigma_\alpha) + \sum_{\beta \in [\alpha]} l(\beta)}{||\alpha|| + 1} \geq \frac{3 ||\alpha|| + 3 ||\alpha||}{||\alpha|| + 1} = \frac{6 ||\alpha||}{||\alpha|| + 1} \geq 4
\]

for each \( \alpha \in C_3 \) with \( ||\alpha|| \geq 2 \).

Since all walks in \( C \) which are not in \( C_3 \) have length at least 4, the average length in \( C \) must also be at least 4.

**Corollary 2.2.** Let \( G \) be a minimally 3-connected graph with \( n \) vertices. If \( C \) is a walk double cover of \( G \) with \( k \) walks, then \( k \leq \frac{3n-4}{2} \).

**Proof.** Let \( m \) denote the number of edges in \( G \). W. Mader proved in [3] that \( m \leq 3n - 6 \); by Theorem 1.1, \( k \leq \frac{m+2}{2} \leq \frac{(3n-6)+2}{2} = \frac{3n-4}{2} \).

**Corollary 2.3.** If \( G \) is a minimally 3-connected planar graph with \( n \) vertices, then \( G \) has at most \( n \) faces. Moreover if \( G \) has exactly \( n \) faces, then \( G \) is a wheel.

**Proof.** Since \( G \) is 3-connected, its set of faces is a walk double cover. By Theorem 1.1, \( m \geq 2r - 2 \), where \( m \) and \( r \) are the number of edges and faces of \( G \), respectively. Since \( n - m + r = 2 \), it follows \( r \leq n \).

Also by Theorem 1.1, if \( G \) is not a wheel, then \( m \geq 2r - 1 \), in which case \( r \leq n - 1 \).

**Corollary 2.4.** If \( G \) is a minimally 3-connected graph with \( n \) vertices embedded in a closed surface \( S \) with Euler characteristic \( \chi \neq 2 \), then \( G \) has at most \( n - \chi \) faces.

**Proof.** As in Corollary 2.3, the set of faces of \( G \) is a walk double cover of \( G \). Since \( S \) is not the sphere, \( C \) is not the set of planar faces of \( G \). By Theorem 1.1, \( m \geq 2r \), where \( m \) and \( r \) are the number of edges and faces of \( G \), respectively. Since \( \chi = n - m + r \), it follows \( r \leq n - \chi \).

**References**

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