ON THE HETEROCHROMATIC NUMBER OF CIRCULANT DIGRAPHS

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Abstract

The heterochromatic number $hc(D)$ of a digraph $D$, is the minimum integer $k$ such that for every partition of $V(D)$ into $k$ classes, there is a cyclic triangle whose three vertices belong to different classes.

For any two integers $s$ and $n$ with $1 \leq s \leq n$, let $D_{n,s}$ be the oriented graph such that $V(D_{n,s})$ is the set of integers mod $2n + 1$ and $A(D_{n,s}) = \{(i, j) : j - i \in \{1, 2, \ldots, n\} \setminus \{s\}\}$.

In this paper we prove that $hc(D_{n,s}) \leq 5$ for $n \geq 7$. The bound is tight since equality holds when $s \in \{n, \frac{2n+1}{3}\}$.

Keywords: circulant tournament, vertex colouring, heterochromatic number, heterochromatic triangle.

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1. Introduction

The heterochromatic number of an $r$-graph $H = (V, E)$ (hypergraph whose edges are sets of size $r$) is the minimum number $k$ such that each vertex colouring of $H$ using exactly $k$ colours leaves at least one edge all whose vertices receive different colours.

The heterochromatic number of $r$-graphs has been studied in several papers including general and particular settings (see for instance [2] – [7]). An important instance of this invariant is the heterochromatic number $hc(D)$ (with respect to $\bar{C}_3$) of a digraph $D$, which is the minimum integer $k$ such that for every partition of $V(D)$ into $k$ classes, there is a cyclic triangle whose
three vertices belong to different classes. The heterochromatic number is preserved under opposition (i.e., $hc(D^{op}) = hc(D)$ where $D^{op}$ denotes the digraph obtained from $D$ by reversing the direction of each arc of $D$).

Let $D_{n,s}$ be the oriented graph such that $V(D_{n,s})$ is the set of integers mod $2n + 1$ and $A(D_{n,s}) = \{(i,j) : j - i \in \{1, 2, \ldots, n\} \setminus \{s\}\}$. In this paper we prove that $hc(D_{n,s}) \le 5$ for $n \ge 7$. The bound is tight since equality holds when $s \in \{n, \frac{2n+1}{3}\}$. Related results concerning the heterochromatic number of circulant tournaments were given in [5] and [7].

2. Preliminaries

For general concepts we refer the reader to [1]. If $D$ is a digraph, $V(D)$ and $A(D)$ (or simply $A$) will denote the sets of vertices and arcs of $D$ respectively. A vertex $k$-colouring of $D$ is said to be full if it uses the $k$ colours. We will denote by $c_1, c_2, \ldots, c_k$ the colours and by $\ell_1, \ell_2, \ldots, \ell_k$ the corresponding chromatic classes. A heterochromatic cyclic triangle (h. triangle) is a cyclic triangle whose vertices are coloured with 3 different colours.

Along this paper we will work in the ring $\mathbb{Z}_{2n+1}$ of integers mod $2n + 1$. If $J$ is a nonempty subset of $\mathbb{Z}_{2n+1} \setminus \{0\}$ such that $|\{j, -j\} \cap J| \le 1$ for every $j \in \mathbb{Z}_{2n+1} \setminus \{0\}$ then the circulant oriented graph $\tilde{C}_{2n+1}(J)$ is defined by $V(\tilde{C}_{2n+1}(J)) = Z_{2n+1}, A(\tilde{C}_{2n+1}(J)) = \{(i,j) : i, j \in Z_{2n+1} \text{ and } j - i \in J\}$ and $\tilde{C}_{2n+1}(J)$ is its underlying graph. In particular, $\tilde{C}_{2n+1} = \tilde{C}_{2n+1}(\{1\})$ is the oriented cycle of length $2n + 1$ and $\tilde{C}_{2n+1}$ is its underlying graph. Finally, for $S \subseteq J_n = \{1, 2, \ldots, n\} \subseteq Z_{2n+1}$, $\tilde{C}_{2n+1}(S)$ will denote the circulant tournament $\tilde{C}_{2n+1}(J)$ where $J = (J_n \cup (-S)) \setminus S$ (when $S = \{s\}$ we will denote $\tilde{C}_{2n+1}(S)$ by $\tilde{C}_{2n+1}(\{s\})$).

The following statement is relevant in our approach.

**Remark.** Given any two different elements $i, j$ of $Z_{2n+1}$, the reflection $\alpha_{i,j}$ of $C_{2n+1}$ defined by $\alpha_{i,j}(x) = i + j - x$ is an antiautomorphism of $\tilde{C}_{2n+1}(J)$ which interchanges $i$ and $j$.

Although the aim of this work is to determine a tight upper bound for $hc(D_{n,s})$, for technical reasons we prefer dealing with $\tilde{C}_{2n+1}(S)$; so we define a normal triangle (n. triangle) of $\tilde{C}_{2n+1}(S)$ to be a cyclic triangle in $\tilde{C}_{2n+1}(S)$ avoiding the arcs of the form $(i + s, i)$, (i.e., a cyclic triangle of $D_{n,s}$).

We will write $(i \in \ell_1 \cup \ell_2, (i,j,k,i))$ to express that we may assume that $i \in \ell_1 \cup \ell_2$ because $(i, j, k, i)$ is an heterochromatic normal triangle (h. n. triangle) whenever $i \notin \ell_1 \cup \ell_2$. 


Let $(j, k)$ be an arc of $\overline{C}_{2n+1}(s)$, along the proofs we will write $(j, k) \sim s$ or $s \sim (j, k)$ (resp. $(j, k) \not\sim s$ or $s \not\sim (j, k)$) to mean that $(j, k) \in \{(i + s, i) \mid i \in \mathbb{Z}_{2n+1}\}$ (resp. $(j, k) \not\in \{(i + s, i) \mid i \in \mathbb{Z}_{2n+1}\}$). For a pair $(j, k)$, we write $s \not\sim (j, k) \in A$ to mean that $(j, k) \not\in A$ and $(j, k) \not\sim s$.

In what follows $\gamma_n(i, j)$ (or simply $\gamma(i, j)$) will denote the $ij$-path $(i, i + 1, \ldots, j)$ (notation mod $2n + 1$) in $C_{2n+1}$ as well as the set of its vertices; $\ell(\gamma(i, j))$ will be the length of $\gamma(i, j)$, i.e., the number of edges of $\gamma(i, j)$.

Two vertex colourings $f$ and $f'$ of a digraph $D$ is said to be equivalent, in symbols: $f \equiv f'$ when there exists either an automorphism or an anti-automorphism $\alpha$ of $D$ such that $f' = f \circ \alpha$. Clearly $\equiv$ is an equivalence relation and $f$ and $f'$ use the same colours whenever $f \equiv f'$.

We will need the following two lemmas.

**Lemma 2.1.** Let $f$ and $f'$ be vertex colourings of $\overline{C}_{2n+1}(s)$.

(i) If $f \equiv f'$ and $f$ leaves an $h$. n. triangle of $\overline{C}_{2n+1}(s)$ then $f'$ leaves an $h$. n. triangle of $\overline{C}_{2n+1}(s)$.

(ii) If $f' = f \circ \alpha_{i,j}$, then $f'(\alpha_{i,j}(x)) = f(x)$. ■

**Lemma 2.2.** Let $f$ be a full vertex $r$-colouring of $C_{2n+1}$.

(i) Suppose $r \geq 4$. If (1) there exist two vertices $a, b \in V(C_{2n+1})$ with $\ell(\gamma(a, b)) = n$ (resp. $n - 1$) such that $a \in \mathcal{E}_2$, $b \in \mathcal{E}_4$, $\mathcal{E}_3 \cap \gamma(a, b) \neq \emptyset$ and $\mathcal{E}_4 \cap \gamma(a, b) \neq \emptyset$, then (2) there exist two vertices $a', b' \in V(C_{2n+1})$ with $\ell(\gamma(a', b')) = n$ (resp. $n - 1$) such that $a' \in \mathcal{E}_i$, $b' \in \mathcal{E}_j$, $\mathcal{E}_k \cap \gamma(b', a') \neq \emptyset$ and $\mathcal{E}_l \cap \gamma(b', a') \neq \emptyset$ for $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$.

(ii) If $r \geq 5$, then (2) holds.

**Proof.** To prove (i), take $c \in \mathcal{E}_3 \cap \gamma(a, b)$ and $d \in \mathcal{E}_4 \cap \gamma(a, b)$, and suppose that $c < d$ ($c$ and $d$ considered as integers).

First consider $b + n$ (resp. $b + n - 1$). Since $\mathcal{E}_2 \cap \gamma(b + n, b) \neq \emptyset$, $\mathcal{E}_3 \cap \gamma(b + n, b) \neq \emptyset$ and $\mathcal{E}_4 \cap \gamma(b + n, b) \neq \emptyset$ we may assume $b + n$ (resp. $b + n - 1$) $\in \mathcal{E}_1$ (in other case we take $a' = b$ and $b' = b + n$ (resp. $b + n - 1$)).

Now, since $c < d$ we have that colours $c_1$, $c_2$ and $c_3$ appear in $\gamma(d + n, d)$; so we may assume $d + n \in \mathcal{E}_4$. Finally we have that colours $c_4$, $c_1$ and $c_2$ appear in $\gamma(c + n, c)$ so we may assume $c + n \in \mathcal{E}_3$ and we obtain the vertices $a, b$ with $c + n \in \mathcal{E}_3 \cap \gamma(b, a)$ and $d + n \in \mathcal{E}_4 \cap \gamma(b, a)$ (resp. $d + n - 1$ and $c + n - 1$).

In order to prove (ii), recall that the number of connected components of $C_{2n+1}\{s\}$ is the maximum common divisor of $s$ and $2n + 1$. In particular,
$C_{2n+1}\{n\}$ is connected and $C_{2n+1}\{n-1\}$ has either 1 or 3 connected components depending on whether $n \not\equiv 1$ or $n \equiv 1 \mod 3$. Since $r = 5$, $C_{2n+1}$ has a vertex $i$ such that $i$ and $i+n$ (resp. $i+n-1$) have different colours. Applying (i) the proof ends.

3. An Upper Bound for $h_c(D_{n,s})$.

In this section we give a tight upper bound for $h_c(D_{n,s})$.

**Theorem 3.1.** For $n \geq 7$, every full vertex 5-colouring of the circulant tournament $C_{2n+1}(s)$ leaves an h. n. triangle; in other words $hc(D_{n,s}) \leq 5$ and equality holds whenever $s \in \{n, \frac{2n+1}{3}\}$.

**Proof.** Consider any full vertex 5-colouring and suppose that no h. n. triangle is produced. We divide the proof into two cases.

**Case 1.** $s \not\equiv n$.

Because of Lemmas 2.2(ii) and 2.1, we may assume that $0 \in \mathcal{C}_1$ and $n+1 \in \mathcal{C}_2$, $\mathcal{C}_3 \cap \gamma(0,n+1) \neq \emptyset$ and $\mathcal{C}_4 \cap \gamma(0,n+1) \neq \emptyset$.

Let $i \in \mathcal{C}_3 \cap \gamma(0,n+1)$ and $j \in \mathcal{C}_4 \cap \gamma(0,n+1)$; we may assume that $|\{(n+1,i),(i,0)\} \cap A| = 1$ and $|\{(n+1,j),(j,0)\} \cap A| = 1$. If $|\{(n+1,i),(i,0)\} \cap A| = 0$, then $(0,i,n+1,0)$ is an h. n. triangle and if $|\{(n+1,i),(i,0)\} \cap A| = 2$, then $(0,j,n+1,0)$ is an h. n. triangle. Similarly $|\{(n+1,j),(j,0)\} \cap A| = 1$. Moreover $|\{(n+1,j),(n+1,i)\} \cap A| = 1$ and $|\{(i,0),(j,0)\} \cap A| = 1$. We may assume w.l.o.g. that $(i,0) \in A$ (with $(i,0) \sim s$) and $(n+1,j) \in A$ (with $(n+1,j) \sim s$). Now observe that when $\mathcal{C}_5 \cap \gamma(0,n+1) \neq \emptyset$, $(0,k,n+1,0)$ is an h. n. triangle, where $k \in \mathcal{C}_5 \cap \gamma(0,n+1)$. So we may assume that $\mathcal{C}_5 \cap \gamma(0,n+1) = \emptyset$ and then $\mathcal{C}_5 \cap \gamma(n+1,0) \neq \emptyset$.

Let $k \in \mathcal{C}_5 \cap \gamma(n+1,0)$. We will analyze several possible cases.

**Subcase 1.a.** $s \not\sim (j,k) \in A$.

$s \sim (0,k) \in A$. In other case $(0,j,k,0)$ is an h. n. triangle ($s \not\sim (0,j) \in A$ as $(i,0) \sim s$).

When $(i,k) \in A$ we have $(i,k) \not\sim s$ (because $(i,0) \sim s$), also we have $2s \geq n+1$ (as $(i,0) \sim s$, $(0,k) \sim s$ and $(i,k) \in A$ with $(i,k) \not\sim s$); so $s > 1$; ($1 \in \mathcal{C}_1 \cup \mathcal{C}_2$, $(0,1,n+1,0)$) (notice $1 \neq s$, $n \neq s$) and then $(i,k,1,i)$ is an h. n. triangle. When $(k,i) \in A$ with $(k,i) \not\sim s$ we have $2s < n$ and hence $i < j$; also we observe that $s \sim (j,i) \in A$ (in other case $(j,k,1,i)$ is an h. n. triangle and $s \sim (k,n+1) \in A$ (otherwise $(k,i,n+1,k)$ is an h. n. triangle;
so we obtain: $3s = n + 1 ((n + 1, j) \sim s, (j, i) \sim s$ and $(i, 0) \sim s)$, $2s = n ((0, k) \sim s$ and $(k, n + 1) \sim s)$, so $s = 1$ and $2n + 1 = 5$ contradicting $n \geq 7$.

When $s \sim (k, i) \in A$ we have $j < i$ (because $(n + 1, j) \sim s$); in this case also we have $2s > n + 1$, so $s > 1$ and $(1 \in \mathcal{C}_1 \cup \mathcal{C}_2, (0, 1, n + 1, 0))$; we conclude that $(j, k, 1, j)$ is an h. n. triangle.

**Subcase 1.b.** $s \sim (j, k) \in A$.

Since $(n + 1, j) \sim s$ and $(j, k) \sim s$ with $k \in \gamma(n + 1, 0)$ we have $2s > n + 1$ and hence $i > j$. Observe $(k, j + 1) \in A$ (because $(j, k) \sim s < n$). Now $n + 1 = i$: when $j + 1 = i$ we get the h. n. triangle $(k, j + 1, j, n + 1, k)$ (Notice that $(n + 1, k) \neq s$ as $(j, k) \sim s$ and $n + 1 \neq j$ since $(n + 1, j) \sim s$). When $j + 1 \neq i$ we obtain $(j + 1 \in \mathcal{C}_1 \cup \mathcal{C}_2, (0, j + 1, n + 1, 0))$, now if $j + 1 \in \mathcal{C}_1$ then $(j + 1, n + 1, k, j + 1)$ is an h. n. triangle (we have observed that $(n + 1, k) \neq s$) and if $j + 1 \in \mathcal{C}_2$ then $(j + 1, n, k, j + 1)$ is an h. n. triangle (notice that $(n, k) \neq s$ because $(j, k) \sim s$ and $j \neq n$ as $j < i \in \gamma(n + 1, 0)$).

Finally, if $s = \frac{2n + 1}{3}$, the vertex 4-colouring defined by $(0 \in \mathcal{C}_1, s \in \mathcal{C}_2, 2s \in \mathcal{C}_3$ and $x \in \mathcal{C}_4$ for $x \not\in \{0, s, 2s\}$) leaves no h. n. triangle and, since $s \neq n$, we obtain $hc(D_{n,s}) = 5$.

**Case 2.** $s = n$.

Because of Lemmas 2.2(ii) and 2.1, we may assume that $n + 2 \in \mathcal{C}_2, 0 \in \mathcal{C}_1$, $\mathcal{C}_3 \cap \gamma(0, n + 2) \neq \emptyset$ and $\mathcal{C}_4 \cap \gamma(0, n + 2) \neq \emptyset$.

For every $x \in \gamma(3, n - 1)$, $x \in \mathcal{C}_1 \cup \mathcal{C}_2$. In other case $(0, x, n + 2, 0)$ is an h. n. triangle.

We may assume: (1) $(\mathcal{C}_3 \cup \mathcal{C}_4) \cap \{1, 2\} \neq \emptyset$ (when $(\mathcal{C}_3 \cup \mathcal{C}_4) \cap \{1, 2\} = \emptyset$ we obtain $(\mathcal{C}_3 \cup \mathcal{C}_4) \cap \{n, n + 1\} \neq \emptyset$ and such a colouring is equivalent to another one satisfying (1) by Lemma 2.1(ii) where $\alpha_{i,j} = \alpha_{0,n+2}$). Suppose $\mathcal{C}_3 \cap \{1, 2, n, n + 1\} = \emptyset$, then $\mathcal{C}_5 \cap \gamma(n + 2, 0) \neq \emptyset$, let $k \in \mathcal{C}_5 \cap \gamma(n + 2, 0)$ and
let \( i \in \{1, 2\} \cap (\mathcal{C}_3 \cup \mathcal{C}_4) \). If \( k \in \gamma(n + 4, 2n - 2) \) or if \((k = 3 \text{ and } i = 1)\) then \((i, n - 1, k, i)\) is an h. n. triangle; now suppose \( k = n + 3 \text{ and } i = 2 \); clearly we may assume \( 1 \in \mathcal{C}_1 \cup \mathcal{C}_2 \) (otherwise \((1, n - 1, n + 3, 1)\) is an h. n. triangle), also we may assume \( n + 1 \in \mathcal{C}_4 \) (otherwise \( n \in \mathcal{C}_4 \) and \( (1, n, n + 3, 1)\) is an h. n. triangle), moreover \( n + 5 \in \mathcal{C}_3 \) (otherwise \((n + 5 \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3, \ (2, n - 1, n + 5, 2)\) and \((n + 5 \in \mathcal{C}_3 \cup \mathcal{C}_4, \ (2, n + 1, n + 5, 2))\)), so \((2, n + 1, n + 5, 2)\) is an h. n. triangle. Hence \( k \in \{2n, 2n - 1\}\) (notice that \( k \neq n + 2 \) as \( n + 2 \in \mathcal{C}_2 \) and \( k \in \mathcal{C}_5\)). If \( i = 2 \) we have \((n + 1 \in \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5, \ (i, n + 1, k, i)\) (notice that \( i \in \mathcal{C}_3 \cup \mathcal{C}_4 \) and \( k \in \mathcal{C}_5\)); when \( n + 1 \in \mathcal{C}_5 \) we are done, so \( n + 1 \in \mathcal{C}_3 \cup \mathcal{C}_4 \) and then \((n + 1, k, 3, n + 1)\) is an h. n. triangle; we conclude that \( i = 1 \) and \( 2 \notin \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5 \), and so \( \{n, n + 1\} \cap (\mathcal{C}_3 \cup \mathcal{C}_4) \neq \emptyset \); moreover, again by Lemma 2.1(ii) we may assume that \( n \notin (\mathcal{C}_3 \cup \mathcal{C}_4) \), \( 1 \in \mathcal{C}_3 \) and \( n + 1 \in \mathcal{C}_4 \) and then \((n + 1, k, 3, n + 1)\) is an h. n. triangle.

Suppose now that \( \mathcal{C}_5 \cap \{1, 2, n, n + 1\} \neq \emptyset \) it follows that there exists an arc \((a, b)\) with \( a \in \{1, 2\}, \ b \in \{n, n + 1\}, \ \ell(\gamma(a, b)) = n - 1 \), \( a \in \mathcal{C}_i \), \( b \in \mathcal{C}_j \) and \( \{i, j\} \in \{3, 4, 5\}\) without loss of generality assume \( 1 \in \mathcal{C}_3 \) and \( n \in \mathcal{C}_4 \) (the other possible cases are completely analogous). Now \((n + 5 \in \mathcal{C}_3 \cup \mathcal{C}_4, \ (1, n, n + 5, 1)\) (remember \( n \geq 7 \)) and \( \{2, n + 1\} \cap \mathcal{C}_5 \neq \emptyset \). When \( 2 \in \mathcal{C}_5 \) we get \((n + 5, 2, n - 1, n + 5)\) an h. n. triangle and when \( n + 1 \in \mathcal{C}_5 \) we obtain the h. n. triangle \((n + 1, n + 5, 3, n + 1)\).

Finally, since the vertex 4-colouring of \( D_{n,n} \) defined by \((0 \in \mathcal{C}_1, n \in \mathcal{C}_2, n + 1 \in \mathcal{C}_3 \text{ and } x \in \mathcal{C}_4 \text{ for } x \notin \{0, n, n + 1\}\) leaves no h. n. triangle, we obtain \( hc(D_{n,n}) = 5 \).

4. Final Comment

It can be proved that \( hc(D_{n,s}) = 4 \) whenever \( n \geq 7 \) and \( s \notin \{n, (2n + 1)/3\} \).

The complete determination of \( hc(D_{n,s}) \), which is a useful tool in studying 4-heterochromatic cycles in circulant tournaments, requires an extensive proof and will be given elsewhere.

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References


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