EQUATIONAL BASES FOR WEAK MONOUNARY VARIETIES

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Abstract

It is well-known that every monounary variety of total algebras has one-element equational basis (see [5]). In my paper I prove that every monounary weak variety has at most 3-element equational basis. I give an example of monounary weak variety having 3-element equational basis, which has no 2-element equational basis.

Keywords: partial algebra, weak equation, weak variety, regular equation, regular weak equational theory, monounary algebras.

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1. Introduction

Weak equations and varieties were studied by H. Höft [4]. An algebraic characterization of weak varieties, under a condition named “conflict free”, is shown in [7]. A completeness theorem for weak equational logic was given by L. Rudak in [6]. G. Bińczak [1] characterized weak varieties as classes closed under homomorphic images and mixed products.

Basic definitions and facts about partial algebras can be found in [3] (Chapter 2) and in [2].

In this section we set up notation and terminology.
Definition 1.1. Let $n$ be a natural number and $A$ a set. A relation $f \subseteq A^n \times A$ is called an $n$-ary partial operation in the set $A$ if and only if for every $a \in A^n$, $b, c \in A$, if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$. If $f$ is an $n$-ary partial operation in a set $A$, then $\text{dom}(f) = \{a \in A^n : \exists b \in A (a, b) \in f\}$.

The notation $f(a) = b$ means that $(a, b) \in f$.

Definition 1.2. A pair $(F, \eta)$ is called a type or (signature) (of algebras), if $F$ is an arbitrary set and $\eta : F \to \omega$. A type $(F, \eta)$ is monounary if and only if a set $F$ has exactly one element $f$ and $\eta(f) = 1$. Let $(F, \eta)$ be a type. Then a pair $\underline{A} = (A, (f^\underline{A})_{f \in F})$ is called partial algebra of type $(F, \eta)$ if and only if $A \neq \emptyset$ and, for every $f \in F$, $f^\underline{A}$ is an $\eta(f)$-ary partial operation in $A$. We call $A$ the support of $\underline{A}$. A partial algebra $\underline{A} = (A, (f^\underline{A})_{f \in F})$ is called monounary if and only if it is of some monounary type.

In the sequel we fix a monounary type $(F, \eta)$ ($F = \{f\}$) and we will consider only partial algebras of this type.

Let $X$ be a countable set of variables. The usual monounary total term algebra generated by a set $X$ will be denoted by $T(X)$. Every monounary term is of the form $x^n = \underbrace{f \ldots f}_n x$, where $x \in X$, $n$ is a natural number and $f$ is an operation symbol. If $n = 0$, then $x^n$ denotes $x$. If $\underline{A} = (A, (f^\underline{A})_{f \in F})$ is a monounary partial algebra and $p = \underbrace{f \ldots f}_n x = x^n$ is a term, then the term operation $p^\underline{A}$ is a monounary partial operation in the set $A$ such that $p^\underline{A}(a) = \underbrace{f(f(\ldots f(a)\ldots))}_{n \text{ times}}$ if $a \in \text{dom}(p^\underline{A})$. The domain of $(x^n)^\underline{A}$ is defined inductively:

$$\text{dom}((x^0)^\underline{A}) = A$$

and

$$\text{dom}((x^n)^\underline{A}) = \{a \in A : a \in \text{dom}((x^{n-1})^\underline{A}) \text{ and } (x^{n-1})^\underline{A}(a) \in \text{dom}(f^\underline{A})\}.$$

If $p, r \in T(X)$, $z \in X$, $p = x^k$ and $r = y^n$ for some $x, y \in X$ and $k, n \in N$, then

$$p(z/r) = \begin{cases} y^{k+n}, & \text{if } z = x, \\ p, & \text{if } z \neq x. \end{cases}$$
A pair of terms \((p, q) \in T(X)^2\) (where \(p = x^k\) and \(q = y^m\) for some \(x, y \in X\) and \(k, m \in \mathbb{N}\)) is a weak equation in a partial algebra \(A\) \((A \models p \approx q)\) iff for every \(a, b \in A\),

if \(x = y\), then \(a \in \text{dom}p^A \cap \text{dom}q^A\) implies \(p^A(a) = q^A(a)\)

and

if \(x \neq y\), then \(a \in \text{dom}(p^A)\) and \(b \in \text{dom}(q^A)\) implies \(p^A(a) = q^A(b)\).

Instead of \((p, q)\) we will write \(p \approx q\).

Let \(E \subseteq T(X)^2\), \(K\) be a class of algebras and \(B \in K\). We write:

\[ K \models p \approx q \text{ iff for every } A \in K, \quad A \models p \approx q, \]

\[ B \models E \text{ iff for every } p \approx q \in E, \quad B \models p \approx q. \]

Let \(\text{Eq}_w(K) = \{(p, q) \in T(X)^2; \ K \models p \approx q\}\) and \(\text{Mod}_w(E) = \{A; \ A \models p \approx q\text{ for every } (p, q) \in E\}\). A class \(K\) of algebras is a weak variety iff \(K = \text{Mod}_w(\text{Eq}_w(K))\). An algebraic characterization of weak varieties is shown in [1].

A set \(I \subseteq T(X)\) is an initial segment iff for every \(x^n \in I\), if \(m, n \in \mathbb{N}\) and \(m < n\) then \(x^m \in I\). A set \(E \subseteq T(X)^2\) is an equational basis of a weak variety \(K\) iff \(\text{Mod}_w(E) = K\). A set \(E \subseteq T(X)^2\) is a weak equational theory iff \(E = \text{Eq}_w(K)\) for some class of algebras \(K\); equivalently (see [6]) it is closed under the following rules:

- **R1** \(p \approx p\) (reflexivity);
- **R2** \(p \approx q, q \approx r \Rightarrow p \approx r\) (symmetry);
- **R3** \(p \approx q, r_1 \approx r_2, \ldots, r_n \approx q\) if \(r_1, \ldots, r_n \in D_E(p, q)\), where \(D_E(p, q)\) is the smallest initial segment \(I \subseteq T(X)\) such that \(X \cup \{p, q\} \subseteq I\); and if \(r \in I\), \(f(s) \in I\) and \(r \approx s \in E\), then \(f(r) \in I\) (weak transitivity);
- **R4** \(p \approx q \Rightarrow f(p) \approx f(q)\);
- **R5** \(p \approx q \Rightarrow p(x/r) \approx q(x/r)\) for some \(x \in X\) and \(r \in T(X)\) (substitution).
Weak equational theory $E \subseteq T(X)^2$ is nontrivial if there exist $p, q \in T(X)$ such that $p \neq q$ and $p \approx q \in E$. If $E_1, E_2 \subseteq T(X)^2$, then $E_1 \vdash E_2$ iff $E_2$ follows from $E_1$ by above rules, equivalently: for every weak equational theory $E$, if $E_1 \subseteq E$, then $E_2 \subseteq E$. If $E$ is a weak equational theory, then a subset $E_0 \subseteq E$ is a basis of $E$ iff $E_0$ is an equational basis of $\text{Mod}_w(E)$, equivalently: $E_0 \vdash E$. Moreover, if $E_1, E_2 \subseteq T(X)^2$, $E_1 \vdash E_2$ and $E_2 \vdash E_1$, then $E_1$ is a basis of $E$ iff $E_2$ is a basis of $E$.

**Definition 1.3.** An equation $p \approx q \in T(X)^2$ is regular if and only if $p = x^n$ and $q = x^m$ for some $x \in X$ and $n, m \in N$. A weak equational theory $E$ is regular if and only if every equation in $E$ is regular.

## 2. Regular weak equational theories

In this section we prove (Corollary 2.10) that every regular weak equational theory has a 2-element basis.

**Lemma 2.1.** Let $E$ be a weak equational theory, $n \geq 1$ and $k \geq 0$. If $x^k \approx x^{k+n} \in E$ and $m \geq k$, then $x^m \approx x^{m+rn} \in E$ for every $r \geq 0$.

**Proof.** By rule R5, $x^k(x^{m-k}) \approx x^{k+n}(x^{m-k}) \in E$, so $x^m \approx x^{m+n} \in E$, which proves lemma for $r = 1$. Suppose that we prove lemma for $r \leq l$. Let $x^k \approx x^{k+n} \in E$ and $m \geq k$. Then $x^m \approx x^{m+ln} \in E$ and $x^{m+ln} \approx x^{m+ln+n} \in E$ (since $m + ln \geq k$). By rule R3, $x^m \approx x^{m+ln+n} \in E$, so $x^m \approx x^{m+(l+1)n} \in E$. 

**Lemma 2.2.** Let $E$ be a weak equational theory, $k, n, m \geq 0$. If $x^k \approx x^{k+n} \in E$ and $x^k \approx x^{k+m} \in E$, then $x^k \approx x^{k+n+m} \in E$.

**Proof.** By Lemma 2.1 $x^{k+m} \approx x^{k+m+n} \in E$ (since $x^k \approx x^{k+n} \in E$ and $k + m \geq k$). Therefore, $x^k \approx x^{k+m} \in E$ and $x^{k+m} \approx x^{k+n+m} \in E$. By rule R3, $x^k \approx x^{k+n+m} \in E$.

**Corollary 2.3.** Let $E$ be a weak equational theory. Let $l \geq 1, k, a_i, n_i \geq 0$ for $1 \leq i \leq l$. If $x^k \approx x^{k+n_1}, \ldots, x^k \approx x^{k+n_l} \in E$, then $x^k \approx x^{k+a_1n_1 + \ldots + a_ln_l} \in E$. 


Lemma 2.4. Let $E$ be a weak equational theory, $n \geq 1$, $k \geq 0$ and $x \in X$. If $x^k \approx x^{k+n} \in E$, $p, q \geq 0$ and $\max(p, q) \geq k + n$, then for every $s \in N$, $x^s \in D_E(x^p, x^q)$.

Proof. Suppose that there exists $s \in N$ such that $x^s \notin D_E(x^p, x^q)$. Let $r = \min\{s \in N : x^s \notin D_E(x^p, x^q)\}$. Since $x = x^0 \in D_E(x^p, x^q)$, we have $r > 0$. Moreover, $r > p$ and $r > q$, since $x^p, x^q \in D_E(x^p, x^q)$ and $D_E(x^p, x^q)$ is an initial segment. Therefore, $r - 1 \geq \max(p, q) \geq k + n$, $r - 1 - n \geq k$ and, by Lemma 2.1,

$$x^{r-1-n} \approx x^{r-1} \in E,$$

since $x^k \approx x^{k+n} \in E$. Moreover, $x^{r-1-n}, x^{r-1} \in D_E(x^p, x^q)$ by definition of $r$ and $f(x^{r-1-n}) = x^{r-n} \in D_E(x^p, x^q)$, since $n \geq 1$. By definition of $D_E(x^p, x^q)$ (cf. [R3]), $f(x^{r-1}) = x^r \in D_E(x^p, x^q)$ and we have a contradiction with definition of $r$. ■

Lemma 2.5. Let $E$ be a nontrivial weak equational theory. If $r, d \geq 1$, $s \geq r$, $k, d_0 \geq 0$, $x^k \approx x^{k+rd} \in E$ and $x^{d_0} \approx x^{d_0+d} \in E$, then $x^k \approx x^{k+sd} \in E$.

Proof. There exists $a \geq 1$ such that $k + ard > d_0$. Then $x^{k+ard} \approx x^{k+ard+sd} \in E$, by Lemma 2.1, since $x^{d_0} \approx x^{d_0+d} \in E$. Moreover, $x^k \approx x^{k+ard}, x^{k+sd} \approx x^{k+sd+ard} \in E$, by Lemma 2.1, since $x^k \approx x^{k+rd} \in E$. Therefore,

$$x^k \approx x^{k+ard}, x^{k+ard} \approx x^{k+ard+sd}, x^{k+ard+sd} \approx x^{k+sd} \in E$$

and, by Lemma 2.4, $x^{k+ard}, x^{k+ard+sd} \in D_E(x^k, x^{k+sd})$, since $x^k \approx x^{k+rd} \in E$ and $\max(k, k + sd) = k + sd \geq k + rd$. Hence, by rule R3, $x^k \approx x^{k+sd} \in E$. ■

Definition 2.6. Let $x \in X$ and $E$ be a nontrivial weak equational theory. Define $R_x(E) = \{n > 0 : \text{there exists } k \geq 0 \text{ such that } x^k \approx x^{k+n} \in E\}$. By rule R5, $R_x(E) = R_y(E)$ for every $x, y \in X$. So, we can write $R(E)$ instead of $R_x(E)$.
Lemma 2.7. Let $E$ be a nontrivial weak equational theory. Then

1. if $n_1, n_2 \in R(E)$, then $n_1 + n_2 \in R(E)$,
2. if $n_1, n_2 \in R(E)$ and $n_1 - n_2 > 0$, then $n_1 - n_2 \in R(E)$,
3. if $n \in R(E)$ and $r \geq 0$, then $rn \in R(E)$.

Proof.

1. If $n_1, n_2 \in R(E)$, then there exist $k_1, k_2 \geq 0$ such that $x^{k_1} \approx x^{k_1+n_1} \in E$ and $x^{k_2} \approx x^{k_2+n_2}$. Let $k = k_1 + k_2$. By Lemma 2.1, $x^{k} \approx x^{k+n_1} \in E$ and $x^{k} \approx x^{k+n_2} \in E$. Hence, $x^{k} \approx x^{k+n_1+n_2} \in E$ by Lemma 2.2. Therefore, $n_1 + n_2 \in R(E)$.

2. If $n_1, n_2 \in R(E)$ and $n_1 - n_2 > 0$, then there exist $k_1, k_2 \geq 0$ such that $x^{k_1} \approx x^{k_1+n_1} \in E$ and $x^{k_2} \approx x^{k_2+n_2}$. Let $k = k_1 + k_2$. By Lemma 2.1, $x^{k} \approx x^{k+n_1} \in E$ and $x^{k} \approx x^{k+n_2} \in E$. Therefore, $x^{k+n_2} \approx x^{k} \in E$ and $x^{k} \approx x^{k+n_1} \in E$. By rule R3, $x^{k+n_2} \approx x^{k+n_1} \in E$. Hence, $x^{k+n_2} \approx x^{k+n_1+n_2} \in E$ and $n_1 - n_2 \in R(E)$.

3. If $n \in R(E)$ and $r \geq 0$, then there exists $k \geq 0$ such that $x^{k} \approx x^{k+n} \in E$. By Lemma 2.1, $x^{k} \approx x^{k+rn} \in E$. Hence $rn \in R(E)$. ■

Corollary 2.8. Let $E$ be a nontrivial weak equational theory. If $a_1, \ldots, a_n \in R(E)$, then $\gcd(a_1, \ldots, a_n) \in R(E)$.

Proof. Let $d = \gcd(a_1, \ldots, a_n)$. Then $d = b_1a_1 + \ldots + b_ia_i + c_{i+1}a_{i+1} + \ldots + c_n a_n$, where $b_j \geq 0$ and $c_j < 0$. By Lemma 2.7, $d_1 = b_1a_1 + \ldots + b_ia_i \in R(E)$ and $d_2 = -(c_{i+1}a_{i+1} + \ldots + c_n a_n) \in R(E)$. Hence, $d = d_1 - d_2 \in R(E)$ by Lemma 2.7. ■

If $E$ is a nontrivial weak equational theory, then a set $R(E)$ is infinite. Suppose that $R(E) = \{a_1, \ldots, a_n, \ldots\}$. Let $d_n = \gcd(a_1, \ldots, a_n)$ for $n \geq 1$. Then we have a sequence $d_1 \geq d_2 \geq \ldots > 0$. Therefore, there exists $n \geq 1$ such that $d_n = d_{n+1} = \ldots$. By Corollary 2.8, $d = d_n \in R(E)$ and $d = \gcd(R(E))$ (i.e. $d|k$ for every $k \in R(E)$ and if there exists $d_0 \geq 1$ such that $d_0|k$ for every $k \in R(E)$, then $d|d_0$). Moreover, $R(E) = \{kd \in N : k > 0 \text{ and } k \in N\}$. 

Lemma 2.9. Let $E$ be a nontrivial weak equational theory. Let $d = \text{gcd}(R(E))$, $x \in X$ and $d_0 = \min\{k \geq 0: x^k \approx x^{k+d} \in E\}$. Let $k_0 = \min\{k \geq 0: \text{there exists } n > k \text{ such that } x^k \approx x^n \in E\}$. Let further $l_0 = \min\{k \geq 0: x^{k_0}\approx x^{k_0+k} \in E\}$ and

$$E_0 = \{x^{k_0} \approx x^{k_0+l_0}, x^{d_0} \approx x^{d_0+d}\}.$$ 

Then $E_0 \subseteq E$ and for every weak equational theory $E'$ such that $E_0 \subseteq E'$ and for every $y^k \approx y^m \in E$, we have $y^k \approx y^m \in E'$.

**Proof.** Let $E'$ be a weak equational theory such that $E_0 \subseteq E'$ and $y^k \approx y^m \in E$. We show that $y^k \approx y^m \in E'$.

By rule R5, $E_0' = \{y^{k_0} \approx y^{k_0+l_0}, y^{d_0} \approx y^{d_0+d}\} \subseteq E' \cap E$.

Suppose that $n > 0$. Then $n \in R(E)$ and $d | n$. Hence, there exists $r \geq 1$ such that $n = rd$.

If $d_0 \leq k$, then $y^k \approx y^{k+rd} \in E'$ by Lemma 2.1, since $y^{d_0} \approx y^{d_0+d} \in E'$.

Suppose that $d_0 > k$. Since $k_0 \leq k$ by definition of $k_0$, we have $k_0 \leq k < d_0$.

We know that $l_0 \in R(E)$. Hence, there exists $r_0 \geq 1$ such that $l_0 = r_0 d$.

Suppose that $r \geq r_0$. Then $y^k \approx y^{k+l_0} \in E'$ by Lemma 2.1, since $y^{k_0} \approx y^{k_0+l_0} \in E'$ and $k_0 \leq k$. Therefore, $y^k \approx y^{k+rd} \in E'$ by Lemma 2.5, since $y^k \approx y^{k+rd} \in E'$, $y^{d_0} \approx y^{d_0+d} \in E'$ and $r \geq r_0$. Hence, $y^k \approx y^{k+n} \in E'$, since $n = rd$.

Suppose that $r_0 > r$. We show that $k + n \geq d_0 + d$.

Suppose that $k + n < d_0 + d$. Then $d_0 - 1 + d \geq k + n$ and $y^{d_0-1+nd} \in D_E(y^{d_0-1},y^{d_0-1+d})$ by Lemma 2.4, since $d_0 - 1 + d \geq k + n$. By Lemma 2.1, we have $y^{d_0-1} \approx y^{d_0-1+nd} \in E$ and $y^{d_0-1+nd} \approx y^{d_0-1+d} \in E$, since $d_0 - 1 \geq k$, $d_0 - 1 + d \geq d_0$, $n - 1 \geq 0$, $y^k \approx y^{k+n} \in E$ and $y^{d_0} \approx y^{d_0+d} \in E$.

We have

$$y^{d_0-1} \approx y^{d_0-1+nd}, y^{d_0-1+nd} \approx y^{d_0-1+d} \in E$$
and \(y^{d_0-1+nd} \in D_E(y^{d_0-1}, y^{d_0-1+d})\). By rule R3, we obtain \(y^{d_0-1} \approx y^{d_0-1+d} \in E\) and we have a contradiction with definition of \(d_0\).

We know that \(y^k \approx y^{k+{}rd} \in E'\), since \(k \geq k_0\) and \(y^{k_0} \approx y^{k_0+{}rd} = y^{k_0} \approx y^{k_0+0} \in E'_0 \subseteq E'\). Moreover, \(k + rd = k + n \geq d_0 + d > d_0\) and \(k + r_0d - (k + rd) = (r_0 - r)d \geq 0\) implies \(y^{k+{}rd} \approx y^{k+{}rd} \in E'\) by Lemma 2.1. We have \(y^{k+{}rd} \in D_{E'}(y^k, y^{k+n})\), by Lemma 2.4, since \(k + n \geq d_0 + d\) and \(y^{d_0} \approx y^{d_0+d} \in E'_0 \subseteq E'\). Therefore,

\[
y^k \approx y^{k+{}rd}, y^{k+{}rd} \approx y^{k+{}rd} \in E', y^{k+{}rd} \in D_{E'}(y^k, y^{k+{}rd})
\]

and, by rule R3, \(y^k \approx y^{k+{}rd} = y^k \approx y^{k+n} \in E'\).

**Corollary 2.10.** Every regular weak equational theory \(E\) has a 2-element basis.

**Proof.** If \(E\) is not nontrivial weak equational theory, then \(E = \{(p, p) \in T(X)^2 : p \in T(X)\}\) and every 2-element subset of \(E\) is a basis of \(E\).

Suppose that \(E\) is nontrivial regular weak equational theory. Let \(d = \gcd(R(E)) \in R(E)\). Fix \(x \in X\). Then the set \(\{k \geq 0 : x^k \approx x^{k+d} \in E\}\) is not empty. Let \(d_0 = \min\{k \geq 0 : x^k \approx x^{k+d} \in E\}\). Let \(k_0 = \min\{k \geq 0 : \text{there exists } n > k \text{ such that } x^k \approx x^n \in E\}\). Let \(l_0 = \min\{k \geq 0 : x^{k_0} \approx x^{k_0+k} \in E\}\). We show that

\[
E_0 = \{x^{k_0} \approx x^{k_0+l_0}, x^{d_0} \approx x^{d_0+d}\}
\]

is a basis of \(E\).

We know that \(E_0 \subseteq E\) by definitions of \(d_0, l_0, k_0\).

Let \(E'\) be a weak equational theory such that \(E_0 \subseteq E'\). We show that \(E \subseteq E'\). Let \(z^k \approx y^m \in E\). Then \(z = y\), since \(E\) is regular and, by Lemma 2.9, we have \(y^k \approx y^m \in E'\). Therefore, \(E \subseteq E'\) and \(E_0\) is a basis of \(E\).

### 3. Main theorem

**Lemma 3.1.** Let \(E\) be a weak equational theory, \(p, q \in N\), \(x, y \in X\) and \(x \neq y\). If \(x^p \approx y^q \in E\), \(p' \geq p\) and \(q' \geq q\), then \(x^{p'} \approx y^{q'} \in E\).
Proof. If \(x^p \approx y^q \in E\), then \(x^p(x/x^p)^{\prime} \approx y^q(x/x^p)^{\prime} = x^{p'} \approx y^q \in E\) by rule R5. Hence, \(x^{p'}(y/y^{p'}) \approx y^q(y/y^{p'}) = x^{p'} \approx y^{p'} \in E\) by rule R5. ■

Lemma 3.2. Let \(E\) be a weak equational theory, \(n \geq 1\), \(x, y \in X\) and \(x \neq y\). Let \(p, q, k \in N\). If \(x^k \approx x^{k+n}\), \(x^p \approx y^q \in E\), then \(x^p \approx y^{k+n} \in E\).

Proof. There exists \(a \geq 1\) such that \(k + an > p\). By Lemma 3.1, \(x^{k+an} \approx y^q \in E\). Hence, \(y^{k+an} \approx y^q \in E\) by rule R5 and \(y^q \approx y^{k+an} \in E\) by rule R2.

By rule R5, \(y^k \approx y^{k+n} \in E\) (since \(x^k \approx x^{k+n} \in E\)). Therefore, \(y^{k+an} \approx y^{k+n} \in E\) by Lemma 2.1, since \(k + n \geq k\), \(k + an = k + n + (a - 1)n\) and \(a - 1 \geq 0\). By rule R2, we have \(y^{k+an} \approx y^{k+n} \in E\).

We have

\[
x^p \approx y^q, y^q \approx y^{k+an}, y^{k+an} \approx y^{k+n} \in E
\]

and \(y^q, y^{k+an} \in D_E(x^p, y^{k+n})\) by Lemma 2.4, since \(y^k \approx y^{k+n} \in E\). Hence, \(x^p \approx y^{k+n} \in E\) by rule R3. ■

Lemma 3.3. Let \(E\) be a weak equational theory, \(x, y \in X\) and \(x \neq y\). If \(m < l\), \(x^m \approx y^l \in E\) and \(x^m \approx y^l \in E\), then \(x^n \approx y^l \in E\).

Proof. By rule R5, \(x^m \approx x^l \in E\). Hence, \(x^n \approx y^l\) by Lemma 3.2, since \(l = m + (l - m)\) and \(l - m \geq 1\). ■

Theorem 3.4. Every monounary weak variety of partial algebras has an at most 3-element equational basis.

Proof. Let \(V\) be a weak monounary variety and \(E = Eq_\omega(V) \subseteq T_F(X)^2\). We show that \(E\) has at most 3-element basis.

If \(E\) is a trivial weak equational theory, then \(E = \{(p, p) \in T(X)^2; p \in T(X)\}\) and every 3-element subset of \(E\) is a basis of \(E\).

If \(E\) is a regular weak equational theory, then \(E\) has a 2-element basis by Corollary 2.10.

Suppose that \(E\) is not regular. Then there exist \(x, y \in X\), \(p, q \in N\) such that \(x^p \approx y^q \in E\) and \(x \neq y\). By Lemma 3.1, \(x^{\max(p,q)} \approx y^{\max(p,q)} \in E\) and the set \(\{n \geq 0; x^n \approx y^n \in E\}\) is not empty.
Let \( m = \min \{ n \geq 0 : x^n \approx y^n \in E \} \) and \( d = \gcd(R(E)) \in R(E) \).

Observe that the set \( \{ k \geq 0 : x^k \approx x^{k+d} \in E \} \) is not empty. Let \( d_0 = \min \{ k \geq 0 : x^k \approx x^{k+d} \in E \} \) and let \( k_0 = \min \{ k \geq 0 : \) there exists \( n > k \) such that \( x^k \approx x^n \in E \} \). Let further \( l_0 = \min \{ k \geq 0 : x^{k_0} \approx x^{k_0+k} \in E \} \), \( k_1 = \min \{ k \geq 0 : \) there exists \( n \geq k \) such that \( x^k \approx y^n \in E \} \) and let \( l_1 = \min \{ k > 0 : x^{k_1} \approx y^{k_1+k} \in E \} \).

We show that

\[
E_1 = \{ x^{k_0} \approx x^{k_0+l_0}, x^{d_0} \approx x^{d_0+l_0}, x^{m} \approx y^{m}, x^{k_1} \approx y^{k_1+l_1} \}
\]

is a basis of \( E \).

Obviously, \( E_1 \subseteq E \) by definitions of \( l_1, d_0, l_0 \).

Let \( E' \) be a weak equational theory such that \( E_1 \subseteq E' \). Let \( z^p \approx t^q \in E \) for some \( z, t \in X \) and \( p, q \geq 0 \). If \( z = t \), then \( z^p \approx t^q \in E' \) by Lemma 2.9. Suppose that \( z \neq t \). By rule R5, \( x^p \approx y^q \in E \). By rule R2 (symmetry), we can assume that \( q \geq p \).

1. If \( q = p \), then \( m \leq q \) and \( x^m \approx y^m \in E_1 \subseteq E' \). Hence, \( x^p \approx y^q \in E' \) by Lemma 3.1 and \( z^p \approx t^q \in E' \) by rule R5.

2. If \( q > p \), then \( x^{k_1} \approx y^q \in E \) by Lemma 3.3, since \( x^{k_1} \approx y^{k_1+l_1} \in E_1 \subseteq E \). Hence, \( k_1 + l_1 \leq q \) by definition of \( l_1 \). Moreover, \( k_1 \leq p \) by definition of \( k_1 \). Therefore, \( x^p \approx y^q \in E' \) by Lemma 3.1, since \( x^{k_1} \approx y^{k_1+l_1} \in E_1 \subseteq E' \), \( k_1 \leq p \) and \( k_1 + l_1 \leq q \). Hence, \( z^p \approx t^q \in E' \) by rule R5.

We proved that \( E \subseteq E' \) and \( E_1 \) is a 4-element basis of \( E \).

Now we show some connections between exponents of equations in \( E_1 \).

By Lemma 3.1, \( x^m \approx y^{m+1} \in E \), since \( x^m \approx y^m \in E_1 \subseteq E \). Hence, \( x^m \approx x^{m+1} \in E \) by rule R5. Therefore, \( 1 \in R(E) \) and \( d = 1 \). Moreover, \( k_1 \leq m \) by definition of \( k_1 \), since \( x^m \approx y^{m+1} \in E \).

We show that \( m \leq k_1 + l_1 \leq m + 1 \) and \( m \leq d_0 + 1 \leq m + 1 \).

a) By Lemma 3.1, \( x^{k_1+l_1} \approx y^{k_1+l_1} \in E \) (since \( x^{k_1} \approx y^{k_1+l_1} \in E \)). Hence, \( m \leq k_1 + l_1 \) by definition of \( m \).
b) By Lemma 3.1, \( x^m \approx y^{m+1} \in E \), since \( x^m \approx y^m \in E \). Hence, \( x^m \approx x^{m+1} \in E \) by rule R5. Therefore, \( x^{k1} \approx y^{k1+l1} \in E \) by Lemma 3.2, since \( x^{k1} \approx y^{k1+l1} \in E \). Hence, \( k1 + l1 \leq m + 1 \) by definition of \( l1 \).

c) We know that \( x^m \approx x^{m+1} \in E \). Hence, \( d0 \leq m \) by definition of \( d0 \), since \( d = 1 \).

d) By Lemma 3.2, \( x^m \approx y^{d0+1} \in E \), since \( x^m \approx y^m \in E \) and \( x^{d0} \approx x^{d0+1} \in E \). By rule R5, \( y^m \approx x^{d0+1} \in E \) and \( x^{d0+1} \approx y^m \in E \) by rule R2. Therefore, \( x^{d0+1} \approx y^{d0+1} \) by Lemma 3.2, since \( x^{d0+1} \approx y^m \in E \) and \( x^{d0} \approx x^{d0+1} \in E \). Hence, \( m \leq d0 + 1 \) by definition of \( m \).

Consider the following cases:

1. \( m = k1 + l1 \). Then \( E1^3 = E1 \setminus \{x^m \approx y^m\} \) is a 3-element basis for \( E \), because \( \{x^{k1} \approx y^{k1+l1}\} \vdash \{x^m \approx y^m\} \) by Lemma 3.1, \( x^{k1} \approx y^{k1+l1} \in E1^3 \) and \( E1^3 \vdash E1 \).

2. \( m + 1 = k1 + l1 \) and \( k1 = m \). Then \( E1^2 = E1 \setminus \{x^{k1} \approx y^{k1+l1}\} \) is a 3-element basis for \( E \), because \( \{x^m \approx y^m\} \vdash x^{k1} \approx y^{k1+l1} \) by Lemma 3.1, \( x^m \approx y^m \in E1^2 \) and \( E1^2 \vdash E1 \).

3. \( m + 1 = k1 + l1 \), \( k1 < m \) and \( m = d0 \). Then \( E1^3 = E1 \setminus \{x^{d0} \approx x^{d0+1}\} \) is a 3-element basis for \( E \), because \( \{x^m \approx y^m\} \vdash \{x^m \approx y^{m+1}\} \vdash \{x^{d0} \approx x^{d0+1}\} \) by Lemma 3.1 and rule R5 \((d0 = m)\), \( x^m \approx y^m \in E1^3 \) and \( E1^3 \vdash E1 \).

4. \( m + 1 = k1 + l1 \), \( k1 < m \) and \( m = d0 + 1 \). We show that \( E1^4 = \{x^{k0} \approx x^{k0+1}, x^{d0} \approx x^{d0+1}, x^{k1} \approx y^m\} \) is a 3-element basis of \( E \). By Lemma 3.1, \( \{x^{k1} \approx y^m\} \vdash \{x^m \approx y^m, x^{k1} \approx y^{k1+l1}\} \), since \( k1 \leq m \) and \( m \leq m + 1 = k1 + l1 \). Hence, \( E1^4 \vdash E1 \). By Lemma 3.2, \( \{x^{d0} \approx x^{d0+1}, x^{k1} \approx y^{k1+l1}\} \vdash \{x^{k1} \approx y^m\} \), since \( m = d0 + 1 \). Hence, \( E1 \vdash E1^4 \) and \( E1^4 \) is a 3-element basis of \( E \).

Example 3.5. The weak monounary variety \( V = \text{Mod}_w(\{x^2 \approx y^2, x^1 \approx y^3, x^0 \approx x^3\}) \) has no 2-element basis.
**Proof.** Define the following monounary algebras (digits denote elements of the support and arrows show how the unique partial 1-ary operation acts):

\[A_1: 0 \quad 1 \downarrow, \quad A_2: 0 \quad \rightarrow \rightarrow 1 \downarrow, \]

\[A_3: 0 \quad \downarrow \downarrow 1 \downarrow, \quad A_4: 0 \quad \rightarrow \rightarrow 1 \rightarrow 0 \rightarrow 2, \]

\[A_5: \quad 2 \downarrow \downarrow \downarrow 0 \quad 1 \rightarrow 2, \quad A_6: \quad 0 \rightarrow \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5.\]

It is easy to see that \(A_1 \in V\) and \(A_4 \in V\). Let \(E = \text{Eq}_{\text{la}}(V)\). Observe that

\[(*) \quad x^0 \approx y^n \notin E \quad \text{for every} \ n \in N,

because \(A_1 \not\models x^0 \approx y^n\) for every \(n \in N\) (since \(x^0(0) = 0 \neq 1 = y^n(1)\)) and \(A_1 \in V\). Hence, any basis of \(E\) cannot contain an equation \(x^0 \approx y^n\) for some \(n \in N\).

Suppose that \(E_0\) is a 2-element basis of \(E\). Consider the following three cases:

1. \(E_0\) has two not regular equation. Then we can assume that \(E_0 = \{x^n \approx y^m, x^k \approx y^l\}\) for some \(x, y \in X, x \neq y\) and \(n, m, k, l \in N\). Moreover \(n, m, k, l \geq 1\) by (\(\ast\)). Therefore, \(A_2 \models E_0\), since \(A_2 \models x^p \approx y^q\) for every \(p, q \geq 1\). Hence, \(A_2 \in V\), since \(E_0\) is a basis of \(E\). But \(A_2 \notin V\), since \(A_2 \not\models x^0 \approx x^3 ((x^0)^{A_2}(0) = 0 \neq 1 = (x^3)^{A_2}(0))\), a contradiction.

2. \(E_0\) has exactly one regular equation. Then we can assume that \(E_0 = \{x^n \approx x^{n+k}, x^m \approx y^{m+l}\}\) for some \(x, y \in X, x \neq y\) and \(n, k, m, l \in N\).
If \( k = 0 \), then \( E_0' = \{ x^n \approx y^{m+l} \} \) is a basis of \( E \), which is impossible by the previous case. Thus \( k \geq 1 \). Moreover, \( m \geq 1 \) by (\( * \)). Then \( A_2 \vDash x^n \approx y^{m+l} \) and \( A_2 \notin V \). Therefore, \( A_2 \not\vDash x^n \approx x^{n+k} \), since \( E_0 \) is a basis of \( E \). But \( A_2 \vDash x^p \approx x^q \) for \( p, q \geq 1 \). Hence \( n = 0 \).

Observe that \( A_1 \in V, A_4 \not\vDash x^0 \approx x^1 ((x^0)_{A_1}(0) = 0 \neq 1 = (x^1)_{A_1}(0)) \) and \( A_4 \vDash x^0 \approx x^2 ((x^0)_{A_1}(0) = 0 \neq 2 = (x^2)_{A_1}(0)) \). Hence \( k \geq 3 \). Then \( A_5 \vDash x^0 \approx x^k \) and \( A_5 \notin V \), since \( A_5 \not\vDash x^1 \approx y^3 ((x^1)_{A_5}(1) = 2 \neq 0 = (y^3)_{A_5}(0)) \). Thus \( A_5 \not\vDash x^m \approx y^{m+l} \), since \( E_0 \) is a basis of \( E \). But \( A_5 \vDash x^p \approx y^q \) for \( p, q \geq 2 \). Hence \( m = 1 \).

Moreover, \( A_4 \in V, A_4 \not\vDash x^1 \approx y^1 ((x^1)_{A_4}(0) = 1 \neq 2 = (y^1)_{A_4}(1)) \) and \( A_4 \vDash x^1 \approx y^2 ((x^1)_{A_4}(0) = 1 \neq 2 = (y^2)_{A_4}(0)) \). Hence \( l \geq 2 \). Therefore, \( E_0 = \{ x^0 \approx x^k, x^1 \approx y^{1+l} \} \), \( k \geq 3 \) and \( l \geq 2 \). Then \( A_6 \vDash E_0 \) and \( A_6 \in V \), since \( E_0 \) is a basis of \( E \). But \( A_6 \notin V \), since \( A_6 \not\vDash x^2 \approx y^2 ((x^2)_{A_6}(0) = 2 \neq 5 = (y^2)_{A_6}(3)) \), a contradiction.

3. \( E_0 \) has two regular equation. Then we can assume that \( E_0 = \{ x^n \approx x^m, x^k \approx x^l \} \) for some \( x \in X \) and \( n, m, k, l \in N \). Therefore, \( A_3 \vDash E_0 \) and \( A_3 \in V \), since \( E_0 \) is a basis of \( E \). But \( A_3 \notin V \), since \( A_3 \not\vDash x^2 \approx y^2 ((x^2)_{A_3}(0) = 0 \neq 1 = (y^2)_{A_3}(1)) \), a contradiction.

From this example we know that there exists a weak monounary variety with 3-element basis, which has no 2-element basis.

References


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