INDUCED ACYCLIC TOURNAMENTS IN RANDOM DIGRAPHS: SHARP CONCENTRATION, THRESHOLDS AND ALGORITHMS\footnote{A preliminary version of parts of this work appeared as an extended abstract in LATIN, 2010, Oaxaca, Mexico.}

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Abstract

Given a simple directed graph $D = (V, A)$, let the size of the largest induced acyclic tournament be denoted by $\text{mat}(D)$. Let $D \in \mathcal{D}(n, p)$ (with $p = p(n)$) be a random instance, obtained by randomly orienting each edge of a random graph drawn from $\mathcal{G}(n, 2p)$. We show that $\text{mat}(D)$ is asymptotically almost surely (a.a.s.) one of only 2 possible values, namely either $b^*$ or $b^* + 1$, where $b^* = \lfloor 2 (\log \frac{r}{n} + 0.5) \rfloor$ and $r = p^{-1}$.

It is also shown that if, asymptotically, $2 (\log \frac{r}{n}) + 1$ is not within a distance of $\omega(n)$ from an integer, then $\text{mat}(D)$ is $\lfloor 2 (\log \frac{r}{n}) + 1 \rfloor$ a.a.s. As a consequence, it is shown that $\text{mat}(D)$ is 1-point concentrated for all $n$ belonging to a subset of positive integers of density 1 if $p$ is independent of $n$. It is also shown that there are functions $p = p(n)$ for which $\text{mat}(D)$ is provably not concentrated in a single value. We also establish thresholds (on $p$) for the existence of induced acyclic tournaments of size $i$ which are sharp for $i = i(n) \to \infty$.

We also analyze a polynomial time heuristic and show that it produces a solution whose size is at least $\log \frac{r}{n} + \Theta(\sqrt{\log \frac{r}{n}})$. Our results are valid as long as $p \geq 1/n$. All of these results also carry over (with some slight changes) to a related model which allows 2-cycles.

Keywords: random digraphs, tournaments, concentration, thresholds, algorithms.

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1. Appendix

1.1. mat(D) versus ω(G)

The following lemma relates the probabilities in the two models D(n, p) and G(n, p) for having, respectively, tournaments and cliques of specific sizes. Its proof is similar to the proof of an analogous relationship involving mas(D) and α(G) (maximum size of an independent set in G) established in [23].

**Lemma 1.1.** For any positive integer b, for a random digraph $D \in \mathcal{D}(n, p)$, $\Pr[\text{mat}(D) \geq b] \geq \Pr[\omega(G) \geq b]$, where $G \in \mathcal{G}(n, p)$.

**Proof.** Given a linear ordering $\sigma$ of vertices of $D$ and a subset $A$ of size $b$, we say that $D[A]$ is consistent with $\sigma$ if for every $\sigma_i, \sigma_j \in A$ with $i < j$, $D[A]$ has the arc $(\sigma_i, \sigma_j)$. 


Let $\tau$ denote an arbitrary but fixed ordering of $V$. Once we fix $\tau$, the spanning subgraph of $D$ formed by arcs of the form $(\tau(i), \tau(j))$ ($i < j$) is having the same distribution as $G(n, p)$. Hence, for any $A$, the event of $D[A]$ being consistent with $\tau$ is equivalent to the event of $A$ inducing a clique in $G(n, p)$. Hence,

$$\Pr(\text{mat}(D) \geq b) = \Pr(\exists A, |A| = b, D[A] \text{ is an acyclic tournament})$$

$$= \Pr(\exists A, |A| = b, \exists \sigma, D[A] \text{ is consistent with } \sigma)$$

$$= \Pr(\exists \sigma, \exists A, |A| = b, D[A] \text{ is consistent with } \sigma)$$

$$\geq \Pr(\exists A, |A| = b, D[A] \text{ is consistent with } \tau)$$

$$= \Pr(\omega(G) \geq b).$$

Hence it is natural that we have a bigger upper bound for $\text{mat}(D)$ than we have for $\omega(G)$.

**Note:** Recall that we first draw an undirected $G \in G(n, 2p)$ and then choose uniformly randomly an orientation of $E(G)$. Hence, for any fixed $A \subseteq V$ of size $b$ with $b = \omega(1)$,

$$\Pr(\text{D}[A] \text{ is an acyclic tournament } | G[A] \text{ induces a clique }) = \frac{b!}{2\binom{b}{2}} = o(1).$$

However, there are so many cliques of size $b$ in $G$ that one of them manages to induce an acyclic tournament.

### 1.2. Proof of Theorem ??

We reduce the NP-complete Maximum Clique problem MC($G, k$) to the MAT($D, k$) problem as follows. Given an instance $(G = (V, E), k)$ of the first problem, compute an instance $f(G) = (G', (V, A), k)$ in polynomial time where

$$A = \{(u, v) : uv \in E, u < v\}.$$  

Clearly, $G'$ is a dag and it is easy to see that a set $V' \subseteq V$ induces a clique in $G$ if and only if $V'$ induces an acyclic tournament in $G'$. This establishes that MAT($D, k$) is NP-hard even if $D$ is restricted to be a dag.

The inapproximability of MAT($D$) follows from the following observation. Note that the reduction $G \to f(G)$ is an $L$-reduction in the sense of [20], since $|f(G)| = |G|$ and $\omega(G) = \text{mat}(G')$. Hence, any inapproximability result on maximum clique in undirected graphs (for example [12, 14]), implies a similar inapproximability for the MAT($D$) problem.
1.3. Proof of Claim ??

Order the vertices of $U$ along a Hamilton path $P$ (if any exists) of $H$. An arc $(u, v) \in A$ is a forward arc if $u$ comes before $v$ in $P$ and is a backward arc otherwise. Since $H$ is acyclic, any arc $(v, u) \in A$ must be a forward arc, since otherwise the segment of $P$ from $u$ to $v$ along with $(v, u)$ forms a cycle in $H$.

Now if there is another Hamilton path $Q$ in $H$, $Q \neq P$, then walking along $P$, consider the first vertex $a$ where $Q$ differs from $P$. Then in the path $Q$, $a$ is visited immediately after some vertex $a'$ that comes after $a$ in $P$. But this implies that $(a', a)$ is a backward arc in $H$ contradicting the observation earlier that $H$ has no backward arc.

1.4. Remaining cases of Theorem ??

For $1/wn \leq p < 1/n$,

$$E[X(n, 4)] = \binom{n}{4} \cdot 4! \cdot p^4 \leq n^4 p^6 \leq (1/n^2) = o(1).$$

Now, an acyclic tournament of size 2 is simply an edge which a.a.s. exists since:

$$Pr[\text{mat}(D) < 2] = Pr[D \text{ is the empty graph}] = (1 - 2p)^\binom{n}{2} \leq e^{-n(n-1)p} = o(1),$$

since $p \geq 1/wn \geq w/n^2$. Hence, when $1/wn \leq p \leq 1/n$, $\text{mat}(D) \in \{2, 3\}$, a.a.s.

For $wn^{-2} \leq p < 1/wn$,

$$E[X(n, 3)] = \binom{n}{3} \cdot 3! \cdot p^3 \leq n^3 p^3 = o(1) \text{ since } np = o(1).$$

The proof for $\text{mat}(D) \geq 2$ is the same as in the previous case, since $n^2p = \omega(1)$, and hence, at least one arc will exist, a.a.s. So when $w/n^2 \leq p \leq 1/wn$, $\text{mat}(D) = 2$, a.a.s.

For $(wn^2)^{-1} \leq p \leq w/n^2$, $E[X(n, 3)] = o(1)$, as in the previous case, and so $\text{mat}(D) = 1$ or $2$, a.a.s. When $p < (wn^2)^{-1}$, $\text{mat}(D) = 1$ since $D$ a.a.s. has no directed edge.

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